ORIGINAL ARTICLE

OPEN ACCESS

Certain Coefficient Inequalities Related with Conical Domain and Associated with Linear q-Difference Operator

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Article History:

Submitted: March 13, 2023
Revised: June 22, 2023
Accepted: June 27, 2023
Published online: June 30, 2023

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ABSTRACT

In our present investigation, we extend the work of [1] and define a new subclass of analytic functions in conic regions using the linear q -difference operator. In [1] certain new classes of analytic functions were introduced and their properties were widely investigated. They derived various subordination conditions and coefficient bounds. In this article we obtain several coefficients bounds and other subordination results for functions belonging to the newly defined class. These coefficient bounds can be used to estimate the area and arc length problems in Geometric Function Theory.

Keywords: Analytic functions, subordination, conic domain, linear q -differential operator.

1 | INTRODUCTION

Let A denotes the class of all function $f(z)$ which are analytic in the open unit disk $E=\{z\in C: \ |z|<1\}$ and normalized by $f(0)=0$ and $f'(0)=1$, so each $f\in A$ has the series expansion of the form:

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

A function $f : E\to C$ is called univalent on E if $f(z_1)\neq f(z_2)$ for all $z_1\neq z_2$, $z_1,z_2\in E$. Let $S\subset A$ be the class of all functions which are univalent in E [2]. Recall $D\subset C$ is said to be a starlike with respect to the point $d_0\in D$ if and only if the line segment joining d_0 to every other point $d\in D$ lies entirely in D , while the set D is said to be convex if and only if it is starlike with respect to each of its points. Navanlinna proved that f maps E onto a starlike domain with respect to $d_0=0$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>0, z\in E,$$

while f maps E onto a convex domain if and only if

$$\Re\left(\frac{\left(zf'(z)\right)'}{f'(z)}\right)>0, z\in E.$$

By S^* and K we mean the subclasses of S composed of starlike and convex functions. In 1991, [3] introduced the class UCV of uniformly convex functions which was extensively studied by Ronning and independently [4], [5]. A more convenient characterization of class UCV was given by Ma and Minda as:

$$f(z) \in \mathrm{UCV} \Leftrightarrow f(z) \in \mathrm{A} \text{ and } \Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in E.$$

In 1999, [6] introduced the class $k-$ uniformly convex functions, $k \geq 0$, denoted by $k-\mathrm{UCV}$ and a related class $k-\mathrm{ST}$ as:

$$f \in k-\mathbf{UCV} \Leftrightarrow zf' \in k-\mathbf{ST}.$$

The class $k-\mathrm{UCV}$ was discussed earlier in [7], see also [6] with same extra restriction and without geometrical interpretation by [6]. In 1985, Nasr et.al, studied a natural extension of classical starlikness in order terminology. We say that a function $f(z) \in \mathrm{A}$ is in the class $\mathrm{S}^*_{k,\gamma}$, $k \geq 0$, $\gamma \in C\backslash\{0\}$, if and only if

$\Re\left\{\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right)\right\} > k\left|\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right)\right|$, $z \in E$. Several author investigated the properties of the class, $\mathrm{S}^*_{k,\gamma}$ and their generalizations in several directions for detail study see [3], [7]–[14]. The convolution or Hadamard product of two functions f and g is denoted by $f * g$ is defined as

$$(f*g)z) = \sum_{n=0}^{\infty} a_n b_n z^n,$$

where $f(z)$ is given by (1) and $g(z) = \sum_{n=2}^{\infty} b_n z^n$, $(z \in E)$. If $f(z)$ and $g(z)$ are analytic in E, we say that $f(z)$ is subordinate to $g(z)$, written as $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, which is analytic in E with $w(0)=0$ and $|w(z)|<1$ such that $f(z) = g(w(z))$. Furthermore, if the function $g(z)$ is univalent in E, then we have the following equivalence, see [2], [15]–[17].

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(E) \subset g(E). \quad z \in E.$$

The operators play a significant role in geometric theory of functions. Many differential and integral operators can be written in term of convolution, for details we refer [18]. It is worth mentioning that the technique of convolution helps researchers in further investigation of geometric properties of analytic functions.

For $t \in R$ and $q > 0$, $q \neq 1$, the number $[t,q]$ is defined in [19] as:

$$[t,q] = \frac{1-q^t}{1-q}, \ [0,q] = 0.$$

For non-negative integer n the q-number shift factorial is defined by

$$[n,q]! = [1,q][2,q][3,q]\ldots[n,q], \quad ([0,q]! = 1).$$

We note that when $q \to 1$, $[n,q]!$ reduces to classical definition of factorial. For $f \in A$, the q-derivative operator or q-difference operator is defined as:

$$\partial_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}, \quad z \in E, z \neq, q \neq 1.$$

It can easily be seen that for $n \in N = \{1,2,3,\dots\}$ and $z \in E$.

$$\partial_q z^n = [n,q]z^{n-1}, \quad \partial_q \left\{\sum_{n=1}^{\infty} \quad a_n z^n\right\} = \sum_{n=1}^{\infty} \quad [n,q]a_n z^{n-1}.$$

The q-generalized Pochhammer symbol for $t \in R$ and $n \in N$ is defined by.

$$[t,q]_n = [t,q][t+1,q][t+2,q]\dots[t+(n-1),q],$$

and for $t > 0$, let q-gamma function is given by.

$$\Gamma_q(t+1) = [t,q]\Gamma_q(t) \quad \text{and } \Gamma_q(1) = 1.$$

Definition [19] For $f \in$ A, let the Ruscheweyh q-difference operator be defined as follows:

$$R_q^{\lambda} f(z) = f(z) * F_{q,\lambda+1}(z), \quad (z \in E, \lambda > -1) \tag{1.11}$$

Where

$$F_{q,\lambda+1}(z) = z + \sum_{n=2}^{\infty} \quad \frac{\Gamma_q(n+\lambda)}{[n-1,q]!\Gamma_q(1+\lambda)} z^n = z + \sum_{n=2}^{\infty} \quad \frac{[\lambda+1,q]_{n-1}}{[n-1,q]!} z^n. \tag{1.12}$$

The symbol "*" stands for Hadamard product (or convolution).

From (1.11) we obtain that

$$R_q^0 f(z) = f(z), \quad R_q^1 f(z) = z\,\partial_q f(z)$$

and

$$R_q^m f(z) = \frac{z\,\partial_q^m (z^{m-1} f(z))}{[m,q]!}, \quad (m \in N).$$

Making use of (1.11) and (1.12), the power series of $R_q^{\lambda} f(z)$ is given by.

$$R_q^{\lambda} f(z) = z + \sum_{n=2}^{\infty} \quad \frac{\Gamma_q(n+\lambda)}{[n-1,q]!\,\Gamma_q(1+\lambda)} a_n z^n$$

$$= z + \sum_{n=2}^{\infty} \quad \frac{[\lambda+1,q]_{n-1}}{[n-1,q]!} a_n z^n$$

$$= z + \sum_{n=2}^{\infty} \quad \psi_{n-1} a_n z^n. \tag{1.13}$$

where

$$\psi_{n-1} = \frac{[\lambda+1,q]_{n-1}}{[n-1,q]!}.$$

Note that

$$\lim_{q \to 1^-} F_{q,\lambda+1}(z) = \frac{z}{(1-z)^{\lambda+1}}$$

and

$$\lim_{q \to 1^-} R_q^{\lambda} f(z) = f(z) * \frac{z}{(1-z)^{\lambda+1}}.$$

It is worth mentioning that $q-$ differential operator reduces to the differential operator defined by [2] in the case when $q \to 1^-$. It is easy to check that

$$z\,\partial\left(F_{q,\lambda+1}(z)\right)=\left(1+\tfrac{[\lambda,q]}{q^{\lambda}}\right)F_{q,\lambda+2}(z)-\tfrac{[\lambda,q]}{q^{\lambda}}F_{q,\lambda+1}(z). \quad (1.14)$$

Making use of (1.11), (1.14) and the properties of Hadamard product, we obtain the following equality.

$$z\,\partial_q\left(R_q^{\lambda}f(z)\right)=\left(1+\tfrac{[\lambda,q]}{q^{\lambda}}\right)R_q^{\lambda+1}f(z)-\tfrac{[\lambda,q]}{q^{\lambda}}R_q^{\lambda}f(z). \quad (1.15)$$

If $q\to 1^{-}$, the equality (1.15) implies.

$$z\left(R_q^{\lambda}f(z)\right)'=(1+\lambda)R^{\lambda+1}f(z)-\lambda R^{\lambda}f(z).$$

which is the well-known recurrent formula for Ruscheweyh differential operator.

Taking motivation from the work of [20]–[22], we introduce new subclasses $k-\mathrm{USCVJ}_q^{\lambda}(\gamma),\gamma\in C\backslash\{0\}$.of analytic functions defined as:

Definition 1: Let $f(z)\in \mathrm{A}$. Then $f(z)$ is in the class $k-\mathrm{USCVJ}_q^{\lambda}(\gamma),\gamma\in C\backslash\{0\}$, if it satisfies the condition.

$$\Re\left\{1+\frac{1}{\gamma}(J(\lambda,\alpha,q,z)-1)\right\}>k\left|\frac{1}{\gamma}(J(\lambda,\alpha,q,z)-1)\right|,\quad z\in E.$$

where

$$J(\lambda,\alpha,q,z)=\frac{z\,\partial_q R_q^{\lambda}f(z)+\alpha z^2\,\partial_q^2 R_q^{\lambda}f(z)}{(1-\alpha)R_q^{\lambda}f(z)+\alpha z\,\partial_q R_q^{\lambda}f(z)}=\frac{z\,\partial_q R_q^{\lambda}g(z)}{R_q^{\lambda}g(z)},$$

with

$$g(z)=\alpha z\,\partial_q f(z)+(1-\alpha)f(z). \quad (1.16)$$

By taking specific values of parameters, we obtain many important subclasses studied by various authors in earlier papers. Here we inlist some of them.

(i) For $\alpha=0$, the class $k-\mathrm{USCVJ}_q^{\lambda}(\gamma)$ reduces into the class $k-\mathrm{USJ}_q^{\lambda}(\gamma)$.

i. For $\alpha=0$, $\lambda=0$, $q\to 1$, and $\gamma=\frac{1}{1-\beta}$, $\beta\in C\backslash\{1\}$, the class $k-\mathrm{USCVJ}_q^{\lambda}(\gamma)$ reduces into the class $\mathrm{SD}(k,\beta)$ and $\alpha=0,\lambda=0$, $q\to 1$, and $\gamma=\frac{2}{1-\beta}$, $\beta\in C\backslash\{1\}$, the class $k-\mathrm{USCVJ}_q^{\lambda}(\gamma)$ reduces into the class $\mathrm{KD}(k,\beta)$, studied by both Shams et.al and Owa et.al [1], [23], [24].

ii. For $\alpha=0$, $k=1$, $\lambda=0$, $q\to 1$, and $\gamma=\frac{1}{1-\beta}$, $\beta\in C\backslash\{1\}$, the class $k-\mathrm{USCVJ}_q^{\lambda}(\gamma)$ reduces into the class $\mathrm{S}_p(\beta)$ and $\alpha=0$, $k=1$, $\lambda=0$, $q\to 1$, and $\gamma=\frac{2}{1-\beta}$, $\beta\in C\backslash\{1\}$, the class $k-\mathrm{USCVJ}_q^{\lambda}(\gamma)$ reduces into the class $\mathrm{K}_p(\beta)$, studied by [25].

iii. For $\alpha=0,\lambda=0$, $q\to 1$, the class $k-\mathrm{USCVJ}_q^{\lambda}(\gamma)$ reduces into the class $\mathrm{K}-\mathrm{ST}$ and $\alpha=1,\lambda=0$, $q\to 1$, the class $k-\mathrm{USCVJ}_q^{\lambda}(\gamma)$ reduces into the class $k-\mathrm{UCV}$, introduced by [8], [26].

iv. For $\alpha=0,k=0$, $\lambda=0$, $q\to 1$, and $\gamma=\frac{1}{1-\beta}$, $\beta\in C\backslash\{1\}$, the class $k-\mathrm{USCVJ}_q^{\lambda}(\gamma)$ reduces into the class $\mathrm{S}^{*}(\beta)$ and $k=0$, $\alpha=0$, $\lambda=0$, $q\to 1$, and $\gamma=\frac{2}{1-\beta}$, $\beta\in C\backslash\{1\}$, the class $k-\mathrm{USCVJ}_q^{\lambda}(\gamma)$ reduces into the class $\mathrm{C}(\beta)$, well-known classes of starlike and convex of order respectively.

Geometric Interpretation

A function $f(z) \in \mathrm{A}$ is in the class $k - \mathrm{USCVJ}_q^\lambda(\gamma)$ if and only if $J(\lambda, \alpha, q, z)$ takes all the values in the conic domain $\Omega_{k,\gamma} = p_{k,\gamma}(E)$ such that.

$$\Omega_{k,\gamma} = \gamma \Omega_k + (1 - \alpha),$$

where

$$\Omega_k = \left\{ u + iv \ : \ u > k\sqrt{(u-1)^2 + v^2} \right\}.$$

Since $p_{k,\gamma}(z)$ is convex univalent, so above definition can be written as

$$J(\lambda, \alpha, q, z) \prec p_{k,\gamma}(z),$$

where

$$p_{k,\gamma}(z) = \begin{cases} \dfrac{1 + (1 - 2\gamma)z}{1 - z}, & \text{for } k = 0, \\ 1 + \dfrac{2\gamma}{\pi^2}\left(log \dfrac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2, & \text{for } k = 1, \\ 1 + \dfrac{2\gamma}{1 - k^2} sinh^2 \left\{ \left(\dfrac{2}{\pi} arccos\, k \right) arctan\, h \sqrt{z} \right\}, & \text{for } 0 < k < 1, \\ 1 + \dfrac{\gamma}{k^2 - 1} sin\left(\dfrac{\pi}{2R(t)} \displaystyle\int_0^{\frac{u(z)}{\sqrt{t}}} \dfrac{1}{\sqrt{1 - x^2}\sqrt{1 - (tx)^2}} dx \right) + \dfrac{\gamma}{1 - k^2}, & \text{for } k > 1. \end{cases}$$

The boundary $\partial\Omega_{k,\gamma}$ of the above set becomes have some nice geometry representing different conic domains for specializing the parameter. For instance, the right half plane when $k = 0$, while a hyperbola when $0 < k < 1$. For $k = 1$ the boundary $\partial\Omega_{k,\gamma}$ becomes a parabola and it is an ellipse when $k > 1$ and in this case and $t \in (0,1)$ is chosen such that $k = cosh(\pi K'(t)/(4K(t)))$. Here $K(t)$ is Legender's complete elliptic integral of first kind and $K'(t) = K(\sqrt{1 - t^2})$ and $K'(t)$ is the complementary integral of $K(t)$ for details see [8], [26], [27]. Moreover, $p_{k,\gamma}(E)$ is convex univalent in E [8], [26]. All of these curves have the vertex at the point $\frac{k+\gamma}{k+1}$

2 | PRELIAMINARIES

In this section we present some important lemmas which will be useful in our current investigation.

Lemma 1. [28] Let $p(z) = \sum_{n=1}^{\infty} p_n z^n \prec F(z) = \sum_{n=1}^{\infty} d_n z^n$ in E. If $F(z)$ is convex univalent in E then

$$|p_n| \leq |d_1|, \quad n \geq 1.$$

Lemma 2 [29]. Let $k \in [0, \infty)$ be fixed and let $p_{k,\gamma}$ be defined (8). If

$$p_{k,\gamma}(z) = 1 + d_1 z + d_2 z^2 + \ldots$$

$$d_1 = \begin{cases} \dfrac{2\gamma A^2}{1 - k^2}, & 0 \leq k < 1 \\ \dfrac{8\gamma}{\pi^2}, & k = 1, \\ \dfrac{\pi^2 \gamma}{4(1 + t)\sqrt{t} K^2(t)(k^2 - 1)}, & k > 1, \end{cases}$$

$$d_2=\begin{cases}\dfrac{A^2+2}{3}d_1, & 0\le k<1\\ \dfrac{2}{3}d_1, & k=1,\\ \dfrac{4K^2(t)(t^2+6t+1)-\pi^2}{24K^2(t)(1+t)\sqrt{t}}d_1, & k>1,\end{cases}$$

where $A=\frac{2\cos^{-1}k}{\pi}$, and $t\in(0,1)$ is chosen such that $k=cosh\left(\frac{\pi K'(t)}{K(t)}\right)$, $K(t)$ is the Legendre's complete elliptic integral of the first kind.

Lemma 3. [3] Let $p(z)=1+\sum_{n=1}^{\infty} c_nz^n\in P$, let $p(z)$ be analytic in E and satisfy $Re\{p(z)\}>0$ for z in E, then the following sharp estimate holds.

$$|c_2-\mu c_1^2|\le 2max\{1,|2\mu-1|\},\quad \forall\mu\in C.$$

3 I MAIN RESULTS

Theorem 1: *Let* $g(z)\in k-\mathrm{USCVJ}_q^\lambda(\gamma)$. *Then*

$$R_q^\lambda g(z)\prec z\,exp\int_0^z \frac{p_{k,\gamma}(w(\xi))-1}{\zeta}d\xi, \tag{3.1}$$

where $w(z)$ *is analytic in* E *with* $w(0)=0$ *and* $|w(z)|<1$. *Moreover, for* $|z|=\rho$, *we have*

$$exp\left(\int_0^1 \frac{p_{k,\gamma}(-\rho)-1}{\rho}d\rho\right)\le\left|\frac{R_q^\lambda g(z)}{z}\right|\le exp\left(\int_0^1 \frac{p_{k,\gamma}(\rho)-1}{\rho}d\rho\right), \tag{3.2}$$

where $g(z)$ is given in (1.16).

Proof : If $g(z)\in k-\mathrm{USCVJ}_q^\lambda(\gamma)$, then using the definition 1, we have

$$\frac{z\,\partial_q R_q^\lambda g(z)}{R_q^\lambda g(z)}-\frac{1}{z}=\frac{p_{k,\gamma}(w(z))-1}{z}.$$

For some function $w(z)$ is analytic in E with $w(0)=0$ and $|w(z)|<1$. Integrating above relation and after some simplification with some properties of subordination, we have.

$$R_q^\lambda g(z)\prec z\,exp\int_0^z \frac{p_{k,\gamma}(w(\xi))-1}{\zeta}d\xi. \tag{3.3}$$

This proves (3.1). Noting that the univalent function $p_{k,\gamma}(z)$ maps the disk $|z|<\rho$ $(0<\rho\le1)$ onto a region which is convex and symmetric with respect to the real axis, we see.

$$p_{k,\gamma}(-\rho|z|)\le\Re\{p_{k,\gamma}(w(\rho z)\}\le p_{k,\gamma}(\rho|z|)\quad(0<\rho\le1,\ z\in E). \tag{3.4}$$

Using (3.3) and (3.4) gives

$$\int_0^1 \frac{p_{k,\gamma}(-\rho|z|)-1}{\rho}d\rho\le\Re\int_0^1 \frac{p_{k,\gamma}\big(w(\rho(z)\big)-1}{\rho}d\rho\le\int_0^1 \frac{p_{k,\gamma}(\rho|z|)-1}{\rho}d\rho,$$

for $z\in E$. Consequently, subordination (14) leads us to

$$\int_0^1 \frac{p_{k,\gamma}(-\rho|z|)-1}{\rho}d\rho\le log\left|\frac{R_q^\lambda g(z)}{z}\right|\le\int_0^1 \frac{p_{k,\gamma}(\rho|z|)-1}{\rho}d\rho$$

$$p_{k,\gamma}(-\rho)\le p_{k,\gamma}(-\rho|z|),p_{k,\gamma}(\rho|z|)\le p_{k,\gamma}(\rho)$$

implies that

$$exp\int_0^1 \frac{p_{k,\gamma}(-\rho)-1}{\rho}d\rho \le \left|\frac{R_q^\lambda g(z)}{z}\right| \le exp\int_0^1 \frac{p_{k,\gamma}(\rho)-1}{\rho}d\rho.$$

This completes the proof.

For $\alpha = 0$, we have the following known result, proved in [23].

Corollary. [21] *Let* $f(z) \in k-\mathrm{USJ}_q^\lambda(\gamma)$. *Then*

$$R_q^\lambda f(z) \prec z\, exp\int_0^z \frac{p_{k,\gamma}(w(\xi))-1}{\zeta}d\xi,$$

where $w(z)$ *is analytic in* E *with* $w(0)=0$ *and* $|w(z)|<1$. *Moreover, for* $|z|=\rho$, *we have*

$$exp\left(\int_0^1 \frac{p_{k,\gamma}(-\rho)-1}{\rho}d\rho\right) \le \left|\frac{R_q^\lambda f(z)}{z}\right| \le exp\left(\int_0^1 \frac{p_{k,\gamma}(\rho)-1}{\rho}d\rho\right).$$

Theorem 2: *If* $f(z) \in k-\mathrm{USCVJ}_q^\lambda(\gamma)$. *Then*

$$|a_2| \le \frac{\delta}{\{1+\alpha q\}\psi_1}, \tag{3.5}$$

and

$$|a_n| \le \frac{\delta}{[n-1,q]!\{1+\alpha q[n-1,q]\}!\psi_{n-1}}\prod_{j=1}^{n-2}\left(1+\frac{\delta}{[j,q]}\right), \qquad \text{for } n=3,4,\ldots. \tag{3.6}$$

where $\delta = \frac{|d_1|}{q}$.

Proof: Let

$$\frac{z\,\partial_q R_q^\lambda f(z)+\alpha z^2\,\partial_q^2 R_q^\lambda f(z)}{(1-\alpha)R_q^\lambda f(z)+\alpha z\,\partial_q R_q^\lambda f(z)} = \frac{z\,\partial_q R_q^\lambda g(z)}{R_q^\lambda g(z)} = p(z). \tag{3.7}$$

where $p(z)$ is analytic in E and $p(0)=1$. Let $p(z) = 1+\sum_{n=1}^{\infty} h_n z^n$ and $R_q^\lambda f(z)$ is given by (1.13). Then (3.7) becomes.

$$z+\sum_{n=2}^{\infty}\{1+\alpha q[n-1,q]\}[n,q]\psi_{n-1}a_n z^n$$

$$=\left(\sum_{n=0}^{\infty} h_n z^n\right)\left\{z+\sum_{n=2}^{\infty}\{1+\alpha q[n-1,q]\}\psi_{n-1}a_n z^n\right\}.$$

Now comparing the coefficients of z^n , we obtain

$$\{1+\alpha q[n-1,q]\}[n,q]\psi_{n-1}a_n$$

$$=\{1+\alpha q[n-1,q]\}\psi_{n-1}a_n+\sum_{j=1}^{n-1}\{1+\alpha q[j-1,q]\}\psi_{j-1}a_j h_{n-j}.$$

which implies

$$a_n = \frac{1}{q[n-1,q]\{1+\alpha q[n-1,q]\}\psi_{n-1}}\sum_{j=1}^{n-1}\{1+\alpha q[j-1,q]\}\psi_{j-1}a_j h_{n-j}.$$

Using the results that $|h_n| \le |d_1|$ given in [27], we have

$$|a_n| \le \frac{|d_1|}{q[n-1,q]\{1+\alpha q[n-1,q]\}\psi_{n-1}}\sum_{j=1}^{n-1}\{1+\alpha q[j-1,q]\}\psi_{j-1}|a_j|.$$

Let us take $\delta = \frac{|d_1|}{q}$, and using lemma2, we have

$$|a_n| \leq \frac{\delta}{[n-1,q]\{1+\alpha q[n-1,q]\}\psi_{n-1}} \sum_{j=1}^{n-1} \{1+\alpha q[j-1,q]\}\psi_{j-1}|a_j|.$$

For $n = 2$, we have

$$|a_2| \leq \frac{\delta}{\{1+\alpha q\}\psi_1},$$

which shows that (3.6) holds for $n = 2$. To prove (3.6) we use principle of mathematical induction, for this, consider the case $n = 3$

$$|a_3| \leq \frac{\delta}{[2,q]\{1+\alpha q[2,q]\}\psi_2}\{1+\{1+\alpha q\}\psi_1|a_2|\}.$$

Using (3.5), we have

$$|a_3| \leq \frac{\delta}{[2,q]\{1+\alpha q[2,q]\}\psi_2}\{1+\delta\}.$$

which shows that 3.6) holds for $n = 3$. Let us assume that (3.6) is true for $n \leq t$, that is,

$$|a_t| \leq \frac{\delta}{[t-1,q]!\{1+\alpha q[t-1,q]\}!\psi_{t-1}} \prod_{j=1}^{t-2} \left(1+\frac{\delta}{[j,q]}\right), \qquad \text{for } n = 3,4,\ldots.$$

Consider

$$|a_{t+1}| \leq \frac{\delta}{[t,q]\{1+\alpha q[t,q]\}\psi_t}\left\{\begin{matrix} 1+\{1+\alpha q\}\psi_1|a_2|+\{1+\alpha q[2]\}\psi_2|a_3| \\ +\{1+\alpha q[3]\}\psi_3|a_4|+\ldots\{1+\alpha q[t-1]\}\psi_{t-1}|a_t| \end{matrix}\right\}$$

$$\leq \frac{\delta}{[t,q]\{1+\alpha q[t,q]\}\psi_t}\left\{\begin{matrix} 1+\delta+\delta\left(1+\frac{\delta}{[1,q]}\right)+\delta\left(1+\frac{\delta}{[1,q]}\right)\left(1+\frac{\delta}{[2,q]}\right) \\ +\ldots\delta\prod_{j=1}^{t-2}\left(1+\frac{\delta}{[j,q]}\right) \end{matrix}\right\}$$

$$= \frac{\delta}{[t,q]!\{1+\alpha q[t,q]\}!\psi_t} \prod_{j=1}^{t-1} \left(1+\frac{\delta}{[j,q]}\right).$$

which proves the assertion of theorem $n = t+1$. Hence (3.6) holds for all n, $n \geq 3$. This completes the proof.

For $\alpha = 0$, we have the following known result, proved in [23].

Corollary: *If* $f(z) \in k - \mathrm{USJ}_q^{\lambda}(\gamma)$. *Then*

$$|a_2| \leq \frac{\delta}{\psi_1},$$

and

$$|a_n| \leq \frac{\delta}{[n-1,q]!\psi_{n-1}} \prod_{j=1}^{n-2} \left(1+\frac{\delta}{[j,q]}\right), \qquad \text{for } n = 3,4,\ldots.$$

Theorem 3: *Let* $0 \leq k < \infty$ *be fixed and let* $f(z) \in k - \mathrm{USCVJ}_q^{\lambda}(\gamma)$ *with the form (1) then for a complex number* μ

$$|a_3 - \mu a_2^2| \leq \frac{d_1}{2q\{1+\alpha q[2,q]\}[2,q]\psi_2} max[1, |2v - 1|], \tag{3.8}$$

where

$$v = \frac{1}{2}\left\{1 - \frac{d_2}{d_1} - \frac{d_1}{q} + \mu \frac{d_1\{1+\alpha q[2,q]\}[2,q]\psi_2}{2q\{1+\alpha q\}^2\psi_1^2}\right\}. \tag{3.9}$$

Proof: Let $f(z) \in k - \mathrm{USCVJ}_q^{\lambda}(\gamma)$, then there exists Schwarz function $w(z)$, with $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{z\,\partial_q R_q^{\lambda} f(z) + \alpha z^2\, \partial_q^2 R_q^{\lambda} f(z)}{(1-\alpha) R_q^{\lambda} f(z) + \alpha z\, \partial_q R_q^{\lambda} f(z)} = \frac{z\,\partial_q R_q^{\lambda} g(z)}{R_q^{\lambda} g(z)} = p_{k,\gamma}(w(z)) \quad z \in E. \tag{3.10}$$

Let $p(z) \in \mathrm{P}$ be a function defined as

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \ldots$$

This gives

$$w(z) = \frac{c_1}{2} z + \frac{1}{2}(c_2 - \frac{c_1^2}{2}) z^2 + \ldots$$

and

$$p_{k,\gamma}(w(z)) = 1 + \frac{d_1 c_1}{2} z + \left\{\frac{d_2 c_1^2}{4} + \frac{1}{2}(c_2 - \frac{c_1^2}{2}) d_1\right\} z^2 + \ldots$$

Using above in (3.10), we obtain

$$a_2 = \frac{d_1 c_1}{2q\{1 + \alpha q\}\psi_1}.$$

and

$$a_3 = \frac{1}{q\{1 + \alpha q[2,q]\}[2,q]\psi_2}\left\{\left\{\frac{d_1 c_2}{2} + \frac{c_1^2}{4}(d_2 - d_1)\right\} + \frac{d_1^2 c_1^2}{4q}\right\}.$$

For any complex number μ and after some calculation we have

$$a_3 - \mu a_2^2 = \frac{d_1}{2q\{1 + \alpha q[2,q]\}[2,q]\psi_2}\{c_2 - v c_1^2\},$$

where

$$v = \frac{1}{2}\left\{1 - \frac{d_2}{d_1} - \frac{d_1}{q} + \mu \frac{d_1\{1 + \alpha q[2,q]\}[2,q]\psi_2}{2q\{1 + \alpha q\}^2\psi_1^2}\right\}.$$

Using a lemma3, we obtain required results.

For $\alpha = 0$, then we have the following known result, proved in [23].

Corollary: *Let* $0 \leq k < \infty$ *be fixed and let* $f(z) \in k - \mathrm{USJ}_q^{\lambda}(\gamma)$ *with the form (1) then for a complex number* μ

$$|a_3 - \mu a_2^2| \leq \frac{d_1}{2q[2,q]\psi_2} max[1, |2v - 1|],$$

where

$$v = \frac{1}{2}\left\{1 - \frac{d_2}{d_1} - \frac{d_1}{q} + \mu \frac{d_1[2,q]\psi_2}{2q\psi_1^2}\right\}.$$

Theorem 4: *If a function* $f(z) \in \mathrm{A}$ *has the form (1) satisfies the condition*

$$\sum_{n=2}^{\infty} \{q[n-1,q](k+1) + |\gamma|\}\{1 + \alpha q[n-1,q]\}|\psi_{n-1}||a_n| \leq |\gamma|, \tag{3.11}$$

then $f(z) \in k - \mathrm{USCVJ}_q^\lambda(\gamma)$.

Proof: Let we note that

$$\left|\frac{z\,\partial_q R_q^\lambda g(z)}{R_q^\lambda g(z)} - 1\right| = \left|\frac{z\,\partial_q R_q^\lambda g(z) - R_q^\lambda g(z)}{R_q^\lambda g(z)}\right| = \left|\frac{\sum_{n=2}^{\infty} q[n-1,q]\{1+\alpha q[n-1,q]\}\psi_{n-1} a_n z^n}{z + \sum_{n=2}^{\infty} \{1+\alpha q[n-1,q]\}\psi_{n-1} a_n z^n}\right|$$

$$\leq \frac{\sum_{n=2}^{\infty} |q[n-1,q]\{1+\alpha q[n-1,q]\}\psi_{n-1}||a_n|}{1 - \sum_{n=2}^{\infty} |\{1+\alpha q[n-1,q]\}\psi_{n-1}||a_n|}.$$

From (3.11) it follows that

$$1 - \sum_{n=2}^{\infty} |\{1+\alpha q[n-1,q]\}\psi_{n-1}||a_n| > 0.$$

To show that $f(z) \in k - \mathrm{USCVJ}_q^\lambda(\gamma)$ it is suffices that

$$\left|\frac{k}{\gamma}\left(\frac{z\,\partial_q R_q^\lambda g(z)}{R_q^\lambda g(z)} - 1\right)\right| - \Re\left\{\frac{1}{\gamma}\left(\frac{z\,\partial_q R_q^\lambda g(z)}{R_q^\lambda g(z)} - 1\right)\right\} \leq 1.$$

As

$$\left|\frac{k}{\gamma}\left(\frac{z\,\partial_q R_q^\lambda g(z)}{R_q^\lambda g(z)} - 1\right)\right| - \Re\left\{\frac{1}{\gamma}\left(\frac{z\,\partial_q R_q^\lambda g(z)}{R_q^\lambda g(z)} - 1\right)\right\}$$

$$\leq \frac{k}{|\gamma|}\left|\frac{z\,\partial_q R_q^\lambda g(z)}{R_q^\lambda g(z)} - 1\right| + \frac{1}{|\gamma|}\left|\frac{z\,\partial_q R_q^\lambda g(z)}{R_q^\lambda g(z)} - 1\right|$$

$$\leq \frac{(k+1)}{|\gamma|}\left|\frac{z\,\partial_q R_q^\lambda g(z)}{R_q^\lambda g(z)} - 1\right|$$

$$\leq \frac{(k+1)}{|\gamma|}\frac{\sum_{n=2}^{\infty} |q[n-1,q]\{1+\alpha q[n-1,q]\}\psi_{n-1}||a_n|}{1 - \sum_{n=2}^{\infty} |\{1+\alpha q[n-1,q]\}\psi_{n-1}||a_n|}$$

$$\leq 1.$$

For $\alpha = 0$, then we have the following known result, proved in [23].

Corollary *[23].* *If a function* $f(z) \in \mathrm{A}$ *has the form (1) satisfies the condition*

$$\sum_{n=2}^{\infty} \{q[n-1,q](k+1) + |\gamma|\}|\psi_{n-1}||a_n| \leq |\gamma|$$

then $f(z) \in k - \mathrm{USJ}_q^\lambda(\gamma)$.

For $q \to 1$, $\lambda = 1$, $\alpha = 0$, $\gamma = 1 - \beta$, $0 \leq \beta < 1$ *, then we have the following known result, proved in [24].*

Corollary [24]. *A function* $f \in A$ *and of the form (1) is in the class* $k - \mathrm{USJ}(1-2\beta)$ *, if it satisfies the condition*

$$\sum_{n=2}^{\infty} \{n(k+1) - (k+\beta)\}|a_n| \leq 1-\beta,$$

where $0 \leq \beta < 1$ *and* $k \geq 0$ *.*

For $\lambda = 1,\ \gamma = 1-\beta,\ 0 \leq \beta < 1$ *and* $k = 0$ *, then we have the following known result, proved in [25].*

Corollary [25]. *A function* $f \in A$ *and of the form (1) is in the class* $0 - \mathrm{USJ}(1-\beta)$ *, if it satisfies the condition*

$$\sum_{n=2}^{\infty} \{n-\beta\}|a_n| \leq 1-\beta, \quad 0 \leq \beta < 1.$$

Theorem 5: *Let* $f(z) \in k - \mathrm{USCVJ}_q^{\lambda}(\gamma)$*. Then* $f(E)$ *contains an open disk of radius.*

$$\frac{q\{1+\alpha q\}[\lambda+1,q]}{2q\{1+\alpha q\}[\lambda+1,q][2,q]! + d_1}.$$

where d_1 *is given by (9).*

Proof: Let $w_0 \neq 0$ be a complex number such that $f(z) \neq w_0$ for $z \in E$. Then

$$f_1(z) = \frac{w_0 f(z)}{w_0 - f(z)} = z + \left(a_2 + \frac{1}{w_0}\right) z^2 + \ldots$$

Since $f_1(z)$ is univalent, so

$$\left|a_2 + \frac{1}{w_0}\right| \leq 2.$$

Now using (16), we have

$$\left|\frac{1}{w_0}\right| \leq \frac{2q\{1+\alpha q\}[\lambda+1,q][2,q]! + d_1}{q\{1+\alpha q\}[\lambda+1,q]},$$

hence we have.

$$|w_0| \geq \frac{q\{1+\alpha q\}[\lambda+1,q]}{2q\{1+\alpha q\}[\lambda+1,q][2,q]! + d_1}.$$

For $\alpha = 0,$ then we have the following known result, proved in [21].

Corollary: *Let* $f(z) \in k - \mathrm{USJ}_q^{\lambda}(\gamma)$*. Then* $f(E)$ *contains an open disk of radius*

$$\frac{q[2,q][2,q]!}{2q[2,q][2,q]! + [\lambda+1,q]d_1}.$$

where d_1 *is given lemma 2.*

4 | CONCLUSION

Using the concept of subordination and theory of quantum calculus, we have systematically studied coefficient estimates and related problems for a new class of analytic functions related conic domain. We also pointed out many new and already existing corollaries by assigning by specializing the Parameters.

Acknowledgment: We are thankful to the editors and anonymous reviewers for their constructive suggestions. All authors contributed equally to this manuscript.

Declaration of Conflicting Interest: The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding: The author(s) received no financial support for the research, authorship, and/or publication of this article.

Data Availability Statement: Data available on request from the authors.

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