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Ambiguities in the X-ray Analysis of Crystal Structure Based on Neutrosophic Fuzzy Rough Aczel-Alsina Aggregation Operators and Their Application in Decision-Making

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ABSTRACT

X-ray cystography, frequently mentioned as X-ray crystal analysis, is a valuable and useful technique used to evaluate the three-dimensional atomic structure of crystalline materials. The major theme of this article is to derive the novel theory of neutrosophic fuzzy rough (NFR) information and its valuable operational laws. Furthermore, we evaluate the Aczel-Alsina operational laws based on the NFR information. Furthermore, we derive the NFR Aczel-Alsina weighted averaging (NFRAAWA) operator, NFR Aczel-Alsina ordered weighted averaging (NFRAAOWA) operator, and NFR Aczel-Alsina hybrid averaging (NFRAAHA) operator. Some suitable properties for the above operators are also invented in detail. Additionally, to find the major principle of the X-ray analysis of crystal structure, we illustrate the procedure of multi-attribute decision-making (MADM) techniques based on the invented operators for NSs. Finally, we draw a comparative analysis between proposed operators with some prevailing techniques to evaluate the supremacy and validity of the derived operators.

Keywords: Neutrosophic fuzzy rough Aczel-Alsina operational laws; Neutrosophic fuzzy rough Aczel-Alsina average aggregation operator

1 | INTRODUCTION

X-ray crystallography, sometimes called X-ray crystal analysis, is a potent scientific method for figuring out the atomic structure of crystalline materials in three dimensions. It is frequently used to comprehend the arrangement of atoms within solids in many different domains, especially in chemistry, bio𝕃ℕy, and materials science. Many peoples have analyzed these kinds of applications in various fields based on crip sets but in the consideration of the classical set theory we have a bundle of problems because of limited opinions, such as zero and one. Therefore, [1] addressed the above problems in the shape of the fuzzy set (FS) because FS has positive information whose range is the unit interval. Many individuals have proposed some aggregation operators [2] similarity measures [3] and different types of structure based on FSs [4]. Furthermore, [5], [6] evaluated the intuitionistic FS (IFS) where the IFS has two different information such as positive and negative grades with the same range and their sum is also from the same range such as unit interval. The importance of IFS is very clear because in many situations we face two types of opinion such as supporting or supporting against, truth or false, and positive or negative which are part of our daily life problems. But we notice that during the election many people have the following possibilities, such as putting a vote in fever, against, abstinence, or neutral, and these kinds of problems cannot be handled from the existing techniques such as FSs and IFSs. Therefore, [7] exposed the Neutrosophic set (NS) where the NS contained the triplet grades such as positive, abstinence, and negative with a strong condition that the sum of the triplet will be contained in [0, 3]. NS is a very effective and dominant technique for depicting unreliable and vague information because of their structure, where some applications of the above two ideas such as IFSs and NSs are as follows, aggregation operators [8] Aczel-Alsina operators [9] distance measures [10] fairly aggregation operators [11] and prioritized interaction operators [12].

To cope with vague and unreliable information, the rough set (RS) is a very dominant technique for evaluating inconsistent and vague data which was exposed by [13] in 1998 which is the modified version of the FS theory. RS is a mathematical framework that deals with uncertain and unreliable information, and we found their applications in many fields, such as artificial intelligence, machine learning, and data mining. Furthermore, the fuzzy RS and the rough FS were invented by [14]. Moreover, [15] modified the intuitionistic fuzzy RS (IRS) which is the modified version of the fuzzy RS, where the IRS covered the grade of positive and negative information which is superior then existing techniques. Furthermore, [16] exposed the rough NS (RNS) which is the combination of RS and NS. Additionally, [17] derived the single-valued neutrosophic rough set and their applications. Further, [18] evaluated the neutrosophic fusion rough sets and their applications. The [19] invented the analysis of smart cities based on single-valued neutrosophic rough sets. Moreover, [20] presented the rough NSs and their applications.

In 1982, [21] presented the technique of a new t-norm and t-conorm, called Aczel-Alsina t-norm and t-conorm based on crip set theory. After developing these norms, many scholars have utilized them in the construction of many kinds of operators based on different ideas, such as averaging operators for IFSs [22] geometric operators for IFSs [23] and prioritized operators for IFSs [24]based on Aczel-Alsina norms. Additionally, weighted operators for neutrosophic Z-number [25] aggregation operators for single-valued NSs [26] weighted aggregation operators for simplified NSs [25] power operators for IFSs [27] averaging operators for IFRSs [28] and prioritized operators for IFRSs [29] based on the Aczel-Alsina operational laws [30]. In the above two paragraphs, we discussed all the possibilities of the existing techniques and their utilizations, but no one can derive the perfect idea of the neutrosophic fuzzy rough set which is more valuable and dominant than the existing techniques. Furthermore, no one can derive yet the theory of Aczel-Alsina operators based on NFRSs. The major contribution of this manuscript is listed below:

1. To derive the novel theory of Neutrosophic fuzzy rough sets and their operational laws.
2. To evaluate the Aczel-Alsina operational laws based on the NFR information.
3. To derive the NFRAAWA operator, NFRAAOWA operator, and NFRAAHA operator. Some suitable properties for the above operators are also invented in detail.

4. To find the major principle of the X-ray analysis of crystal structure, we illustrate the procedure of MADM techniques based on the invented operators for NSs.
5. To draw a comparative analysis between proposed operators with some prevailing techniques to evaluate the supremacy and validity of the derived operators.

This manuscript is arranged in the shape: In Section 2, we discussed the NFS, RS, and their operational laws. In Section 3, we derived the novel theory of NFR information and its valuable operational laws. Furthermore, we evaluated the Aczel-Alsina operational laws based on the NFR information. In Section 4, we derived the NFRAAWA operator, NFRAAOWA operator, and NFRAAHA operator. Some suitable properties for the above operators are also invented in detail. In Section 5, we find the major principle of the X-ray analysis of crystal structure, we illustrate the procedure of MADM techniques based on the invented operators for NSs. In Section 6, we draw a comparative analysis between proposed operators with some prevailing techniques to evaluate the supremacy and validity of the derived operators. Some possible concluding remarks are stated in Section 7.

2 | LITERATURE REVIEW

This section aims to revise the theory of NS, RS, and their related operators and operational laws.

Definition 1: [7] A NFS is listed below:

$$\mathfrak{F} = \{(\varkappa, \mathbb{d}(\varkappa), \mathbb{i}(\varkappa), \mathbb{b}(\varkappa) | \varkappa \in p\}$$

The major prominent condition of NFS is as follows: $0 \leq \mathbb{d}(\varkappa) + \mathbb{i}(\varkappa) + \mathbb{b}(\varkappa) \leq 3$, where the positive, abstinence and negative grades are as follows: $\mathbb{d}(\varkappa), \mathbb{i}(\varkappa)$, and $\mathbb{b}(\varkappa)$. The general shape of neutral grade is as follows: $\mathfrak{F} = \{(\mathbb{d}_{\mathfrak{F}}, \mathbb{i}_{\mathfrak{F}}, \mathbb{b}_{\mathfrak{F}})\}$.

Definition 2: [30] For any two norms $\mathcal{H}$ and $\mathfrak{R}$, then the general shape of the Aczel-Alsina (AA) product and sum are stated below:

$$\mathbb{N} \otimes \mathbb{R} = \{(\varkappa, \mathcal{H}\{\mathbb{d}_{\mathbb{N}}(\varkappa), \mathbb{d}_{\mathbb{R}}(\varkappa)\}, \mathfrak{R}\{\mathbb{i}_{\mathbb{N}}(\varkappa), \mathbb{i}_{\mathbb{R}}(\varkappa)\}, \mathfrak{R}\{\mathbb{b}_{\mathbb{N}}(\varkappa), \mathbb{b}_{\mathbb{R}}(\varkappa)\} | \varkappa \in p)\}$$

$$\mathbb{N} \oplus \mathbb{R} = \{(\varkappa, \mathfrak{R}\{\mathbb{d}_{\mathbb{N}}(\varkappa), \mathbb{d}_{\mathbb{R}}(\varkappa)\}, \mathcal{H}\{\mathbb{i}_{\mathbb{N}}(\varkappa), \mathbb{i}_{\mathfrak{R}}(\varkappa)\}, \mathcal{H}\{\mathbb{b}_{\mathbb{N}}(\varkappa), \mathbb{b}_{\mathfrak{R}}(\varkappa)\} | \varkappa \in p)\}$$

Definition 3: [30]Based on the above ideas, the particular shape of AA operational laws are as follows:

$$\mathfrak{F}_1 \oplus \mathfrak{F}_2 = \left\{ \left(\left(\begin{array}{c} \left(1 - \mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\mathbb{d}_{\mathfrak{F}_1}\right)\right)^{\equiv} + \left(-\mathbb{LN}\left(1-\mathbb{d}_{\mathfrak{F}_2}\right)\right)^{\equiv} \right)^{\frac{1}{\equiv}}} \right), \\ \left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\mathbb{i}_{\mathfrak{F}_1}\right)\right)^{\equiv} + \left(-\mathbb{LN}\left(\mathbb{i}_{\mathfrak{F}_2}\right)\right)^{\equiv} \right)^{\frac{1}{\equiv}}} \right), \\ \left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\mathbb{b}_{\mathfrak{F}_1}\right)\right)^{\equiv} + \left(-\mathbb{LN}\left(\mathbb{b}_{\mathfrak{F}_2}\right)\right)^{\equiv} \right)^{\frac{1}{\equiv}}} \right) \end{array} \right) \right) \right\}$$

$$\mathfrak{F}_1 \otimes \mathfrak{F}_2 = \left\{\left(\left(\begin{array}{c}\left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}(\mathbb{d}_{\mathfrak{F}_1})\right)^{\equiv}+\left(-\mathbb{LN}(\mathbb{d}_{\mathfrak{F}_2})\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}(1-\mathbb{i}_{\mathfrak{F}_1})\right)^{\equiv}+\left(-\mathbb{LN}(1-\mathbb{i}_{\mathfrak{F}_2})\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}(1-\mathbb{b}_{\mathfrak{F}_1})\right)^{\equiv}+\left(-\mathbb{LN}(1-\mathbb{b}_{\mathfrak{F}_2})\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right)\right)\right\}$$

$$\tau\mathfrak{F} = \left\{\left(\left(\left(1-\mathbb{E}xp^{-\left(\tau\left(-\mathbb{LN}(1-\mathbb{d}_{\mathfrak{F}})\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \left(\mathbb{E}xp^{-\left(\tau\left(-\mathbb{LN}(\mathbb{i}_{\mathfrak{F}})\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \left(\mathbb{E}xp^{-\left(\tau\left(-\mathbb{LN}(\mathbb{i}_{\mathfrak{F}})\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\right)\right)\right\}$$

$$\mathfrak{F}^{\tau} = \left\{\left(\left(\left(\mathbb{E}xp^{-\left(\tau\left(-\mathbb{LN}(\mathbb{d}_{\mathfrak{F}})\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \left(1-\mathbb{E}xp^{-(\tau\left(-\mathbb{LN}(1-\mathbb{i}_{\mathfrak{F}})^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \left(1-\mathbb{E}xp^{-(\tau\left(-\mathbb{LN}(1-\mathbb{b}_{\mathfrak{F}})^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\right)\right)\right\}$$

Definition 4: [30] Some aggregation operators are listed below for any NFSs, such as

$$NFAAWA_{\omega}\left(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,,,\ldots,\mathfrak{F}_i\right) = \oplus_{i=1}^{n}\left(\omega_i\mathfrak{F}_i\right) = \left\{\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega_i\left(-\mathbb{LN}\left(1-\mathbb{d}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega_i\left(-\mathbb{LN}\left(\mathbb{i}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega_i\left(-\mathbb{LN}\left(\mathbb{b}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right\}$$

Called the neutrosophic fuzzy AA weighted averaging (NFAAWA) operator.

Definition 5: [30] The neutrosophic fuzzy AA ordered weighted averaging (NFAAOWA) operator is derived below:

$$NFAAOWA_{\omega^*}\left(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,,,\ldots,\mathfrak{F}_i\right) = \oplus_{i=1}^{n}\left({\omega^*}_i\mathfrak{F}_{O_{(i)}}\right) = \left\{\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\sum_{i=1}^{n}{\omega^*}_i\left(-\mathbb{LN}\left(1-\mathbb{d}_{\mathfrak{F}_{O_{(i)}}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n}{\omega^*}_i\left(-\mathbb{LN}\left(\mathbb{i}_{\mathfrak{F}_{O_{(i)}}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n}{\omega^*}_i\left(-\mathbb{LN}\left(\mathbb{b}_{\mathfrak{F}_{O_{(i)}}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right\}$$

Where, $\mathfrak{F}_{O(i-1)} \geq \mathfrak{F}_{O(i)}$.

Definition 6: [30] The neutrosophic fuzzy AA hybrid average (NFAAHA) operator is invented by:

$$NFAAHA_{\omega,\omega^*}(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,,\ldots,\mathfrak{F}_i) = \oplus_{i=1}^{n}\left(\omega^*{}_i\mathfrak{F}_{O_{(i)}}\right) = \left\{\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega^*{}_i\left(-\mathbb{LN}\left(1-\mathbb{d}_{\mathfrak{F}_{O_{(i)}}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right) \\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega^*{}_i\left(-\mathbb{LN}\left(\mathbb{i}_{\mathfrak{F}_{O_{(i)}}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega^*{}_i\left(-\mathbb{LN}\left(\mathbb{b}_{\mathfrak{F}_{O_{(i)}}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right\}$$

Definition 7: [13] The lower approximation (LRA) and upper approximation (URA) of $\mathcal{M}$ for the duplet (p,δ) are exposed below:

$$\underbrace{\delta}(\mathcal{M}) = \{\langle(\varkappa \in p: \delta^*(\varkappa) \subseteq \mathcal{M})\rangle\}$$

$$\overbrace{\delta}(\mathcal{M}) = \{\langle(\varkappa \in p: \delta^*(\varkappa) \sqcap \mathcal{M} \neq \emptyset)\rangle\}$$

Where $p = universal\ set$ and $\delta \subseteq p \times p$ with a function $\delta^*: p \to P_{(p)}$, such as $\delta^*(a) = \{y \in p: (x,y) \in \delta\ and\ x \in p\}$, thus (p,δ) is an approximation space with $\mathcal{M} \subseteq p$. The duplet $(\underbrace{\delta}(\mathcal{M}), \overbrace{\delta}(\mathcal{M}))$, called RS if $\underbrace{\delta}(\mathcal{M}) \neq \overbrace{\delta}(\mathcal{M})$.

3 | NFRS: NEUTROSOPHIC ROUGH SETS

In this section, we invented the NFRSs and their operational laws based on the proposed NFRSs. Further, we also simplify some results for the proposed techniques.

Definition 8: Consider a neutrosophic relation $\delta \in NFS_{(p\times p)}$, the duplet (δ,p) represents the neutrosophic fuzzy approximation space with $\mathcal{M} \subseteq p$ and the LRA and URA of $\mathcal{M}$ for the pair (p,δ) are evaluated below:

$$\underbrace{\delta}(\mathcal{M}) = \left\{\langle\left(\varkappa: \underbrace{\mathbb{d}_{\delta}}_{\mathcal{M}}(\varkappa), \underbrace{\mathbb{i}_{\delta}}_{\mathcal{M}}(\varkappa), \underbrace{\mathbb{b}_{\delta}}_{\mathcal{M}}(\varkappa)\right)|\varkappa \in p\rangle\right\}$$

$$\overbrace{\delta}(\mathcal{M}) = \left\{\langle\left(\varkappa: \overbrace{\mathbb{d}}_{\delta_{\mathcal{M}}}(\varkappa), \overbrace{\mathbb{i}}_{\delta_{\mathcal{M}}}(\varkappa), \overbrace{\mathbb{b}}_{\delta_{\mathcal{M}}}(\varkappa)\right)|\varkappa \in p\rangle\right\}$$

Noticed that $\underbrace{\mathbb{d}_{\delta}}_{\mathcal{M}}(\varkappa) = \wedge_{z\in p}[\mathbb{d}_{\delta}(\varkappa,z)\wedge\mathbb{d}_{\mathcal{M}}(z)]$, $\underbrace{\mathbb{i}_{\delta}}_{\mathcal{M}}(\varkappa) = \vee_{z\in p}[\mathbb{i}_{\delta}(\varkappa,z)\vee\mathbb{i}_{\mathcal{M}}(z)]$ and $\underbrace{\mathbb{b}_{\delta}}_{\mathcal{M}}(\varkappa) = \vee_{z\in p}[\mathbb{b}_{\delta}(\varkappa,z)\vee\mathbb{b}_{\mathcal{M}}(z)]$ and $\overbrace{\mathbb{d}}_{\delta_{\mathcal{M}}}(\varkappa) = \vee_{z\in p}[\mathbb{d}_{\delta}(\varkappa,z)\vee\mathbb{d}_{\mathcal{M}}(z)]$, $\overbrace{\mathbb{i}}_{\delta_{\mathcal{M}}}(\varkappa) = \wedge_{z\in p}[\mathbb{i}_{\delta}(\varkappa,z)\wedge\mathbb{i}_{\mathcal{M}}(z)]$ and $\overbrace{\mathbb{b}}_{\delta_{\mathcal{M}}}(\varkappa) = \wedge_{z\in p}[\mathbb{b}_{\delta}(\varkappa,z)\wedge\mathbb{b}_{\mathcal{M}}(z)]$ with $0 \leq \underbrace{\mathbb{d}_{\delta}}_{\mathcal{M}}(\varkappa) + \underbrace{\mathbb{i}_{\delta}}_{\mathcal{M}}(\varkappa) + \underbrace{\mathbb{b}_{\delta}}_{\mathcal{M}}(\varkappa) \leq 3, 0 \leq \overbrace{\mathbb{d}}_{\delta_{\mathcal{M}}}(\varkappa) + \overbrace{\mathbb{i}}_{\delta_{\mathcal{M}}}(\varkappa) + \overbrace{\mathbb{b}}_{\delta_{\mathcal{M}}}(\varkappa) \leq 3$. Further, $\underbrace{\delta}(\mathcal{M})$ and $\overbrace{\delta}(\mathcal{M})$ represents the NFRs with a function $\underbrace{\delta}(\mathcal{M}), \overbrace{\delta}(\mathcal{M}): NFS_{(p)\to}NFS_{(p)}$, called LR and UR approximation operators. Finally, $\delta(\mathcal{M}) = \left(\underbrace{\delta}(\mathcal{M}), \overbrace{\delta}(\mathcal{M})\right) = \left\{\varkappa: \left(\underbrace{\mathbb{d}_{\delta}}_{\mathcal{M}}(\varkappa), \underbrace{\mathbb{i}_{\delta}}_{\mathcal{M}}(\varkappa), \underbrace{\mathbb{b}_{\delta}}_{\mathcal{M}}(\varkappa)\right), \left(\overbrace{\mathbb{d}}_{\delta_{\mathcal{M}}}(\varkappa), \overbrace{\mathbb{i}}_{\delta_{\mathcal{M}}}(\varkappa), \overbrace{\mathbb{b}}_{\delta_{\mathcal{M}}}(\varkappa)\right)|\varkappa \in p\right\}$ represents the NFRS with $\mathbb{d}(\mathcal{M}) = \left(\underbrace{\delta}(\mathcal{M}), \overbrace{\delta}(\mathcal{M})\right) = \left\{\varkappa: \left(\underbrace{\mathbb{d}_{\delta}}_{\mathcal{M}}(\varkappa), \underbrace{\mathbb{i}_{\delta}}_{\mathcal{M}}(\varkappa), \underbrace{\mathbb{b}_{\delta}}_{\mathcal{M}}(\varkappa)\right), \left(\overbrace{\mathbb{d}}_{\delta_{\mathcal{M}}}(\varkappa), \overbrace{\mathbb{i}}_{\delta_{\mathcal{M}}}(\varkappa), \overbrace{\mathbb{b}}_{\delta_{\mathcal{M}}}(\varkappa)\right)\varkappa \in p\right\}$, with a simple shape: $\delta(\mathcal{M}) = \left\{\left(\underbrace{\mathbb{d}}_{\mathfrak{F}}, \underbrace{\mathbb{i}}_{\mathfrak{F}}, \underbrace{\mathbb{b}}_{\mathfrak{F}}\right), \left(\overbrace{\mathbb{d}}_{\mathfrak{F}}, \overbrace{\mathbb{i}}_{\mathfrak{F}}, \overbrace{\mathbb{b}}_{\mathfrak{F}}\right)\right\}$, also called NFR numbers (NFRNs).

Definition 9: The idea of score and accuracy values are listed below for any two NFRNs, such as

$$Sc(\mathfrak{F}) = \frac{1}{3}\left(\underbrace{\mathbb{d}}_{\mathfrak{F}} + \overbrace{\mathbb{d}}_{\mathfrak{F}} - \underbrace{\mathbb{i}}_{\mathfrak{F}} - \overbrace{\mathbb{i}}_{\mathfrak{F}} - \underbrace{\mathbb{b}}_{\mathfrak{F}} - \overbrace{\mathbb{b}}_{\mathfrak{F}}\right), Sc(\mathfrak{F}) \in [0,1]$$

$$Ac(\mathfrak{F}) = \frac{1}{3}\left(\underbrace{\mathbb{d}}_{\mathfrak{F}} + \overbrace{\mathbb{d}}_{\mathfrak{F}} + \underbrace{\mathbb{i}}_{\mathfrak{F}} + \overbrace{\mathbb{i}}_{\mathfrak{F}} + \underbrace{\mathbb{b}}_{\mathfrak{F}} + \overbrace{\mathbb{b}}_{\mathfrak{F}}\right), Ac(\mathfrak{F}) \in [0,1].$$

With some prominent characteristics, such as

If $Sc(\mathfrak{F}_1) > Sc(\mathfrak{F}_2)$ then $\mathfrak{F}_1 > \mathfrak{F}_2$,

If $Sc(\mathfrak{F}_1) < Sc(\mathfrak{F}_2)$ then $\mathfrak{F}_1 < \mathfrak{F}_2$,

If $Sc(\mathfrak{F}_1) = Sc(\mathfrak{F}_2)$ then

If $Ac(\mathfrak{F}_1) > Ac(\mathfrak{F}_2)$ then $\mathfrak{F}_1 > \mathfrak{F}_2$,

If $Ac(\mathfrak{F}_1) < Ac(\mathfrak{F}_2)$ then $\mathfrak{F}_1 < \mathfrak{F}_2$,

If $Ac(\mathfrak{F}_1) = Ac(\mathfrak{F}_2)$ then $\mathfrak{F}_1 = \mathfrak{F}_2$,

The general shape of AA norms is listed below for any two NFRNs, such as

$$\mathbb{N} \otimes \mathbb{R} = \left\{\begin{pmatrix} (\varkappa, \mathcal{H}\left\{\underbrace{\mathbb{d}}_{\mathbb{N}}(\varkappa), \underbrace{\mathbb{d}}_{\mathbb{R}}(\varkappa)\right\}, \Re\left\{\underbrace{\mathbb{i}}_{\mathbb{N}}(\varkappa), \underbrace{\mathbb{i}}_{\mathbb{R}}(\varkappa)\right\}, \Re\left\{\left(\underbrace{\mathbb{b}}_{\mathbb{N}}(\varkappa), \underbrace{\mathbb{b}}_{\mathbb{R}}(\varkappa)\varkappa\right)\right\}, \\ (\varkappa, \mathcal{H}\left\{\overbrace{\mathbb{d}}_{\mathbb{N}}(\varkappa), \overbrace{\mathbb{d}}_{\mathbb{R}}(\varkappa)\right\}, \Re\left\{\overbrace{\mathbb{i}}_{\mathbb{N}}(\varkappa), \overbrace{\mathbb{i}}_{\mathbb{R}}(\varkappa)\right\}, \Re\left\{\left(\overbrace{\mathbb{b}}_{\mathbb{N}}(\varkappa), \overbrace{\mathbb{b}}_{\mathbb{R}}(\varkappa)\varkappa\right)\right\} | \varkappa \in p) \end{pmatrix}\right\}$$

$$\mathbb{N} \oplus \mathbb{R} = \left\{\begin{pmatrix} (\varkappa, \Re\left\{\underbrace{\mathbb{d}}_{\mathbb{N}}(\varkappa), \underbrace{\mathbb{d}}_{\mathbb{R}}(\varkappa)\right\}, \mathcal{H}\left\{\underbrace{\mathbb{i}}_{\mathbb{N}}(\varkappa), \underbrace{\mathbb{i}}_{\Re}(\varkappa)\right\}, \mathcal{H}\left\{\underbrace{\mathbb{b}}_{\mathbb{N}}(\varkappa), \underbrace{\mathbb{b}}_{\Re}(\varkappa)\right\}), \\ (\varkappa, \Re\left\{\overbrace{\mathbb{d}}_{\mathbb{N}}(\varkappa), \overbrace{\mathbb{d}}_{\mathbb{R}}(\varkappa)\right\}, \mathcal{H}\left\{\overbrace{\mathbb{i}}_{\mathbb{N}}(\varkappa), \overbrace{\mathbb{i}}_{\Re}(\varkappa)\right\}, \mathcal{H}\left\{\overbrace{\mathbb{b}}_{\mathbb{N}}(\varkappa), \overbrace{\mathbb{b}}_{\Re}(\varkappa)\right\} | \varkappa \in p \end{pmatrix}\right\}$$

Definition 10: Some AA operational laws for any two NFRNs, such as

$$\mathfrak{F}_1 \oplus \mathfrak{F}_2 = \left\{\begin{pmatrix} \begin{pmatrix} \left(1 - \mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv} + \left(-\mathbb{LN}\left(1-\underbrace{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\underbrace{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv} + \left(-\mathbb{LN}\left(\underbrace{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\mathbb{b}_{\mathfrak{F}_1}\right)\right)^{\equiv} + \left(-\mathbb{LN}\left(\mathbb{b}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right) \end{pmatrix}, \\ \begin{pmatrix} \left(1 - \mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\overbrace{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv} + \left(-\mathbb{LN}\left(1-\overbrace{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\overbrace{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv} + \left(-\mathbb{LN}\left(\overbrace{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\overbrace{\mathbb{b}}_{\mathfrak{F}_1}\right)\right)^{\equiv} + \left(-\mathbb{LN}\left(\overbrace{\mathbb{b}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right) \end{pmatrix} \end{pmatrix}\right\},$$

$$\mathfrak{F}_1 \otimes \mathfrak{F}_2 = \left\{ \begin{pmatrix} \left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \\ \left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{b}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{b}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right) \end{pmatrix}, \begin{pmatrix} \left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\widehat{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\widehat{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\widehat{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\widehat{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \\ \left(\left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\widehat{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\widehat{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right)\right) \end{pmatrix} \right\},$$

$$\tau\mathfrak{F} = \left\{ \begin{pmatrix} \left(1-\mathbb{E}xp^{-\left(\tau\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{d}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \left(\mathbb{E}xp^{-\left(\tau\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{i}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \\ \left(\mathbb{E}xp^{-\left(\tau\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{b}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right) \end{pmatrix}, \begin{pmatrix} \left(1-\mathbb{E}xp^{-\left(\tau\left(-\mathbb{LN}\left(1-\widehat{\mathbb{d}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \left(\mathbb{E}xp^{-\left(\tau\left(-\mathbb{LN}\left(\widehat{\mathbb{i}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \\ \left(\mathbb{E}xp^{-\left(\tau\left(-\mathbb{LN}\left(\widehat{\mathbb{b}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right) \end{pmatrix} \right\},$$

$$\mathfrak{F}^{\tau} = \left\{ \begin{pmatrix} \left(\mathbb{E}xp^{-\left(\tau\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{d}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \left(1-\mathbb{E}xp^{-\left(\tau\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{i}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \\ \left(1-\mathbb{E}xp^{-\left(\tau\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{b}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right) \end{pmatrix}, \begin{pmatrix} \left(\mathbb{E}xp^{-\left(\tau\left(-\mathbb{LN}\left(\widehat{\mathbb{d}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \left(1-\mathbb{E}xp^{-\left(\tau\left(-\mathbb{LN}\left(1-\widehat{\mathbb{i}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \\ \left(1-\mathbb{E}xp^{-\left(\tau\left(-\mathbb{LN}\left(1-\widehat{\mathbb{b}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right) \end{pmatrix} \right\}.$$

Furthermore, we simplify the above operational laws based on some suitable examples, for this, we consider some NFRNS, such as $\mathfrak{F} = \{(0.2,0.3,0.7),(0.4,0.5,0.6)\}, \mathfrak{F}_1 = \{(0.2,0.5,0.3),(0.4,0.6,0.2)\}$ and $\mathfrak{F}_2 = \{(0.7,0.3,0.1),(0.5,0.2,0.8)\}$ with $\equiv = 2$ and $\tau = 4$, thus

$$\mathfrak{F}_1 \oplus \mathfrak{F}_2 = \left\{ \begin{array}{c} \left(\begin{array}{c} \left(1 - \mathbb{E}xp^{-((-\mathbb{LN}(1-0.2))^2 + (-\mathbb{LN}(1-0.7))^2)^{\frac{1}{2}}}\right), \left(\mathbb{E}xp^{-((-\mathbb{LN}(0.5))^2 + (-\mathbb{LN}(0.3))^2)^{\frac{1}{2}}}\right), \\ \left(\mathbb{E}xp^{-((-\mathbb{LN}(0.3))^2 + (-\mathbb{LN}(0.1))^2)^{\frac{1}{2}}}\right) \end{array} \right), \\ \left(\begin{array}{c} \left(1 - \mathbb{E}xp^{-((-\mathbb{LN}(1-0.4))^2 + (-\mathbb{LN}(1-0.5))^2)^{\frac{1}{2}}}\right), \left(\mathbb{E}xp^{-((-\mathbb{LN}(0.6))^2 + (-\mathbb{LN}(0.2))^2)^{\frac{1}{2}}}\right), \\ \left(\mathbb{E}xp^{-((-\mathbb{LN}\,0.2)^2 + (-\mathbb{LN}(0.8))^2)^{\frac{1}{2}}}\right) \end{array} \right) \end{array} \right\}$$

$$= \{(0.706089, 0.249263, 0.074396), (0.577279, 0.184786, 0.196944)\}$$

$$\mathfrak{F}_1 \otimes \mathfrak{F}_2 = \left\{ \begin{array}{c} \left(\begin{array}{c} \left(\mathbb{E}xp^{-((-\mathbb{LN}(0.2))^2 + (-\mathbb{LN}(0.7))^2)^{\frac{1}{2}}}\right), \left(1 - \mathbb{E}xp^{-((-\mathbb{LN}(1-0.5))^2 + (-\mathbb{LN}(1-0.3))^2)^{\frac{1}{2}}}\right), \\ \left(1 - \mathbb{E}xp^{-((-\mathbb{LN}(1-0.3))^2 + (-\mathbb{LN}(1-0.1))^2)^{\frac{1}{2}}}\right) \end{array} \right), \\ \left(\begin{array}{c} \left(\mathbb{E}xp^{-((-\mathbb{LN}(0.4))^2 + (-\mathbb{LN}(0.5))^2)^{\frac{1}{2}}}\right), \left(1 - \mathbb{E}xp^{-((-\mathbb{LN}(1-0.6))^2 + (-\mathbb{LN}(1-0.2))^2)^{\frac{1}{2}}}\right), \\ \left(1 - \mathbb{E}xp^{-((-\mathbb{LN}(1-0.2))^2 + (-\mathbb{LN}(1-0.8))^2)^{\frac{1}{2}}}\right) \end{array} \right) \end{array} \right\}$$

$$= \{(0.192341, 0.541379, 0.310584), (0.316976, 0.184786, 0.196944)\}$$

$$4\mathfrak{F} = \left\{ \begin{array}{c} \left(\begin{array}{c} \left(1 - \mathbb{E}xp^{-(4(-\mathbb{LN}(1-0.2))^2)^{\frac{1}{2}}}\right), \left(\mathbb{E}xp^{-(4(-\mathbb{LN}(0.3))^2)^{\frac{1}{2}}}\right), \\ \left(\mathbb{E}xp^{-(4(-\mathbb{LN}(0.7))^2)^{\frac{1}{2}}}\right) \end{array} \right), \\ \left(\begin{array}{c} \left(1 - \mathbb{E}xp^{-(4(-\mathbb{LN}(1-0.4))^2)^{\frac{1}{2}}}\right), \left(\mathbb{E}xp^{-(4(-\mathbb{LN}(0.5))^2)^{\frac{1}{2}}}\right), \\ \left(\mathbb{E}xp^{-(4(-\mathbb{LN}(0.6))^2)^{\frac{1}{2}}}\right) \end{array} \right) \end{array} \right\} = \{(0.36, 0.09, 0.49), (0.64, 0.25, 0.36)\}$$

$$\mathfrak{F}^4 = \left\{ \begin{array}{c} \left(\begin{array}{c} \left(\mathbb{E}xp^{-(4(-\mathbb{LN}(0.2))^2)^{\frac{1}{2}}}\right), \left(1 - \mathbb{E}xp^{-(4(-\mathbb{LN}(1-0.3))^2)^{\frac{1}{2}}}\right), \\ \left(1 - \mathbb{E}xp^{-(4(-\mathbb{LN}(1-0.7))^2)^{\frac{1}{2}}}\right) \end{array} \right), \\ \left(\left(\mathbb{E}xp^{-(4(-\mathbb{LN}(0.4))^2)^{\frac{1}{2}}}\right), \left(1 - \mathbb{E}xp^{-(4(-\mathbb{LN}(1-0.5))^2)^{\frac{1}{2}}}\right), \left(1 - \mathbb{E}xp^{-(4(-\mathbb{LN}(1-0.6))^2)^{\frac{1}{2}}}\right) \right) \end{array} \right\}$$

$$= \{(0.04, 0.51, 0.91), (0.16, 0.75, 0.84)\}$$

Theorem 1: Consider any three NFRNs, such as $\mathfrak{F} = \left\{ \left(\underbrace{\mathbb{d}}_{\mathfrak{F}}, \underbrace{\mathbb{i}}_{\mathfrak{F}}, \underbrace{\mathbb{b}}_{\mathfrak{F}} \right), \left(\overbrace{\mathbb{d}}_{\mathfrak{F}}, \overbrace{\mathbb{i}}_{\mathfrak{F}}, \overbrace{\mathbb{b}}_{\mathfrak{F}} \right) \right\}$, $\mathfrak{F}_1 = \left\{ \left(\underbrace{\mathbb{d}}_{\mathfrak{F}_1}, \underbrace{\mathbb{i}}_{\mathfrak{F}_1}, \underbrace{\mathbb{b}}_{\mathfrak{F}_1} \right), \left(\overbrace{\mathbb{d}}_{\mathfrak{F}_1}, \overbrace{\mathbb{i}}_{\mathfrak{F}_1}, \overbrace{\mathbb{b}}_{\mathfrak{F}_1} \right) \right\}$, $and\ \mathfrak{F}_2 = \left\{ \left(\underbrace{\mathbb{d}}_{\mathfrak{F}_2}, \underbrace{\mathbb{i}}_{\mathfrak{F}_2}, \underbrace{\mathbb{b}}_{\mathfrak{F}_2} \right), \left(\overbrace{\mathbb{d}}_{\mathfrak{F}_2}, \overbrace{\mathbb{i}}_{\mathfrak{F}_2}, \overbrace{\mathbb{b}}_{\mathfrak{F}_2} \right) \right\}$. Then

$$\mathfrak{F}_1 \oplus \mathfrak{F}_2 = \mathfrak{F}_2 \oplus \mathfrak{F}_1$$

$$\mathfrak{F}_1 \otimes \mathfrak{F}_2 = \mathfrak{F}_2 \otimes \mathfrak{F}_1$$

$$\tau(\mathfrak{F}_1 \oplus \mathfrak{F}_2) = \tau\mathfrak{F}_1 \oplus \tau\mathfrak{F}_2, \tau > 0$$

$$(\tau_1 + \tau_2)\mathfrak{F} = \tau_1\mathfrak{F} + \tau_2\mathfrak{F}, \tau_1, \tau_2 > 0$$

$$(\mathfrak{F}_1 \otimes \mathfrak{F}_2)^{\tau} = \mathfrak{F}_1{}^{\tau} \otimes \mathfrak{F}_2{}^{\tau}, \tau > 0$$

$$\mathfrak{F}^{\tau_1} \otimes \mathfrak{F}^{\tau_2} = \mathfrak{F}^{(\tau_1 + \tau_2)}$$

Proof:

Assume that

$$\mathfrak{F}_1 \oplus \mathfrak{F}_2 = \left(\begin{array}{c}\left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\underbrace{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\underbrace{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\mathbb{b}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\mathbb{b}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right), \\ \left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\overbrace{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\overbrace{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\overbrace{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\overbrace{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\overbrace{\mathbb{b}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\overbrace{\mathbb{b}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right)\end{array}\right)$$

$$= \left(\begin{array}{c}\left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\underbrace{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\underbrace{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\overbrace{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\overbrace{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right), \\ \left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\overbrace{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\overbrace{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\overbrace{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\overbrace{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\overbrace{\mathbb{b}}_{\mathfrak{F}_2}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\overbrace{\mathbb{b}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right)\end{array}\right)$$

$$= \mathfrak{F}_2 \oplus \mathfrak{F}_1 .$$

Consider that

$$\mathfrak{F}_1 \otimes \mathfrak{F}_2 = \left(\begin{array}{c}\left(\begin{array}{c}\left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\underbrace{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\underbrace{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{b}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{b}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right), \\ \left(\begin{array}{c}\left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\overbrace{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\overbrace{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\overbrace{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\overbrace{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\overbrace{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\overbrace{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right)\end{array}\right)$$

$$= \left(\begin{array}{c} \left(\left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\underbrace{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\underbrace{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \\ \left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{b}}_{\mathfrak{F}_2}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{b}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right) \right), \\ \left(\left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\widehat{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\widehat{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\widehat{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\widehat{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \\ \left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\widehat{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\widehat{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right) \right) \end{array} \right) = \mathfrak{F}_2 \otimes \mathfrak{F}_1$$

Let

$$\tau(\mathfrak{F}_1 \oplus \mathfrak{F}_2) = \tau \left(\begin{array}{c} \left(\left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\underbrace{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\underbrace{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \\ \left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}(\mathbb{b}_{\mathfrak{F}_1})\right)^{\equiv}+\left(-\mathbb{LN}(\mathbb{b}_{\mathfrak{F}_2})\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right) \right), \\ \left(\left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\widehat{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\widehat{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\widehat{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\widehat{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \\ \left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\widehat{\mathbb{b}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\widehat{\mathbb{b}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right) \right) \end{array} \right)$$

$$
= \left(\begin{array}{c}
\left(\begin{array}{c}
\left(1 - \mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)} \right), \left(\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)} \right), \\
\left(\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(\mathbb{b}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\mathbb{b}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)} \right)
\end{array} \right), \\
\left(\begin{array}{c}
\left(1 - \mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(1-\overset{\frown}{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\overset{\frown}{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)} \right), \left(\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)} \right), \\
\left(\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{b}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{b}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)} \right)
\end{array} \right)
\end{array} \right)
$$

$$
= \left(\begin{array}{c}
\left(\begin{array}{c}
\left(1 - \mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)} \right), \left(\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)} \right), \\
\left(\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(\mathbb{b}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)} \right)
\end{array} \right), \\
\left(\begin{array}{c}
\left(1 - \mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(1-\overset{\frown}{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)} \right), \left(\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)} \right), \\
\left(\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{b}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)} \right)
\end{array} \right)
\end{array} \right)
$$

$$
\oplus \left(\begin{array}{c}
\left(\begin{array}{c}
\left(1 - \mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)\right)^{\frac{1}{\equiv}}} \right), \left(\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)\right)^{\frac{1}{\equiv}}} \right), \\
\left(\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(\mathbb{b}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)\right)^{\frac{1}{\equiv}}} \right)
\end{array} \right), \\
\left(\begin{array}{c}
\left(1 - \mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(1-\overset{\frown}{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)\right)^{\frac{1}{\equiv}}} \right), \left(\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)\right)^{\frac{1}{\equiv}}} \right), \\
\left(\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{b}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)\right)^{\frac{1}{\equiv}}} \right)
\end{array} \right)
\end{array} \right) = \tau\mathfrak{F}_1 \oplus \tau\mathfrak{F}_2.
$$

Suppose

$$\tau_1\mathfrak{F}\oplus\tau_2\mathfrak{F}=\left\{\begin{array}{c}\left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\tau_1\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{d}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\tau_1\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{i}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\tau_1\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{b}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right),\\ \left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\tau_1\left(-\mathbb{LN}\left(1-\overset{\frown}{\mathbb{d}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\tau_1\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{i}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\tau_1\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{b}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right)\end{array}\right\}$$

$$\oplus\left\{\begin{array}{c}\left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\tau_2\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{d}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\tau_2\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{i}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\tau_2\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{b}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right),\\ \left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\tau_2\left(-\mathbb{LN}\left(1-\overset{\frown}{\mathbb{d}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\tau_2\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{i}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\tau_2\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{b}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right)\end{array}\right\}$$

$$=\left\{\begin{array}{c}\left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left((\tau_1+\tau_2)\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{d}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left((\tau_1+\tau_2)\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{i}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left((\tau_1+\tau_2)\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{b}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right),\\ \left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left((\tau_1+\tau_2)\left(-\mathbb{LN}\left(1-\overset{\frown}{\mathbb{d}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left((\tau_1+\tau_2)\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{i}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left((\tau_1+\tau_2)\tau_1+\tau_2\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{b}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right)\end{array}\right\}=(\tau_1+\tau_2)\mathfrak{F}.$$

Let

$$(\mathfrak{F}_1 \otimes \mathfrak{F}_2)^{\tau} = \left(\begin{array}{c} \left(\begin{array}{c} \left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\underbrace{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\underbrace{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{b}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{b}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right) \end{array}\right), \\ \left(\begin{array}{c} \left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\widehat{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\widehat{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\widehat{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\widehat{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(1-\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1-\widehat{\mathbb{f}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\widehat{\mathbb{f}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right) \end{array}\right) \end{array}\right)^{\tau}$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left(\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(\underbrace{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\underbrace{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right), \left(1-\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right), \\ \left(1-\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{b}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{b}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right) \end{array}\right), \\ \left(\begin{array}{c} \left(\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(\widehat{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(\widehat{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right), \left(1-\mathbb{E}xp^{\left(-\tau\left(\left(-\mathbb{LN}\left(1-\widehat{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\widehat{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right), \\ \left(1-\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(1-\widehat{\mathbb{f}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(-\mathbb{LN}\left(1-\widehat{\mathbb{f}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right) \end{array}\right) \end{array}\right)$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left(\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(\underbrace{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right), \left(1-\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right), \\ \left(1-\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{b}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right) \end{array}\right), \\ \left(\begin{array}{c} \left(\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(\widehat{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right), \left(1-\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(1-\widehat{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right), \\ \left(1-\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(1-\widehat{\mathbb{f}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right) \end{array}\right) \end{array}\right)$$

$$\otimes\left(\begin{array}{c}\left(\begin{array}{c}\left(\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right),\left(1-\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right),\\ \left(1-\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{b}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right)\end{array}\right),\\ \left(\begin{array}{c}\left(\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right),\left(1-\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(1-\overset{\frown}{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right),\\ \left(1-\mathbb{E}xp^{-\left(\tau\left(\left(-\mathbb{LN}\left(1-\overset{\frown}{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right)\end{array}\right)\end{array}\right)$$

$$=\mathfrak{F}_1{}^{\tau}\otimes\mathfrak{F}_2{}^{\tau}.$$

Consider

$$\mathfrak{F}^{\tau_1}\otimes\mathfrak{F}^{\tau_2}=\left(\begin{array}{c}\left(\begin{array}{c}\left(\mathbb{E}xp^{-\left(\tau_1\left(\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{d}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right),\left(1-\mathbb{E}xp^{-\left(\tau_1\left(\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{i}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right),\\ \left(1-\mathbb{E}xp^{-\left(\tau_1\left(\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{b}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right)\end{array}\right),\\ \left(\begin{array}{c}\left(\mathbb{E}xp^{-\left(\tau_1\left(\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{d}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right),\left(1-\mathbb{E}xp^{-\left(\tau_1\left(\left(-\mathbb{LN}\left(1-\overset{\frown}{\mathbb{i}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right),\\ \left(1-\mathbb{E}xp^{-\left(\tau_1\left(\left(-\mathbb{LN}\left(1-\overset{\frown}{\mathbb{i}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right)\end{array}\right)\end{array}\right)$$

$$\otimes\left(\begin{array}{c}\left(\left(\mathbb{E}xp^{-\left(\tau_2\left(\left(-\mathbb{LN}\left(\underset{\sim}{\mathbb{d}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right),\left(1-\mathbb{E}xp^{-\left(\tau_2\left(\left(-\mathbb{LN}\left(1-\underset{\sim}{\mathbb{i}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right),\left(1-\mathbb{E}xp^{-\left(\tau_2\left(\left(-\mathbb{LN}\left(1-\underset{\sim}{\mathbb{b}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right)\right),\\ \left(\left(\mathbb{E}xp^{-\left(\tau_2\left(\left(-\mathbb{LN}\left(\widehat{\mathbb{d}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right),\left(1-\mathbb{E}xp^{-\left(\tau_2\left(\left(-\mathbb{LN}\left(1-\widehat{\mathbb{i}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right),\left(1-\mathbb{E}xp^{-\left(\tau_2\left(\left(-\mathbb{LN}\left(1-\widehat{\mathbb{i}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right)\right)\end{array}\right)$$

$$=\left(\begin{array}{c}\left(\left(\mathbb{E}xp^{-\left((\tau_1+\tau_2)\left(\left(-\mathbb{LN}\left(\underset{\sim}{\mathbb{d}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right),\left(1-\mathbb{E}xp^{-\left((\tau_1+\tau_2)\left(\left(-\mathbb{LN}\left(1-\underset{\sim}{\mathbb{i}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right),\left(1-\mathbb{E}xp^{-\left((\tau_1+\tau_2)\left(\left(-\mathbb{LN}\left(1-\underset{\sim}{\mathbb{b}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right)\right),\\ \left(\left(\mathbb{E}xp^{-\left((\tau_1+\tau_2)\left(\left(-\mathbb{LN}\left(\widehat{\mathbb{d}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right),\left(1-\mathbb{E}xp^{-\left((\tau_1+\tau_2)\left(\left(-\mathbb{LN}\left(1-\widehat{\mathbb{i}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right),\left(1-\mathbb{E}xp^{-\left((\tau_1+\tau_2)\left(\left(-\mathbb{LN}\left(1-\widehat{\mathbb{b}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right)\right)\end{array}\right)$$

$$=\mathfrak{F}^{(\tau_1+\tau_2)}.$$

4 | NFR-AA AVERAGING AGGREGATION OPERATORS

This section proposed the idea of the NFRAAWA operator, NFRAAOWA operator, and NFRAAHA operator, and their related results or properties.

Definition 11: The NFRAAWA operator is denoted and defined below:

$$NFRAAWA:(\Xi^{\iota})\rightarrow\Xi^{\iota}$$

$$NFRAAWA(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,\ldots\ldots,\mathfrak{F}_n)=\oplus_{i=1}^{n}(\omega_i\mathfrak{F}_i)=\omega_1\mathfrak{F}_1\oplus\omega_2\mathfrak{F}_2\oplus\ldots\oplus\omega_n\mathfrak{F}_n$$

Theorem 2: To consider the data in the above information, we derive that our obtained result will be also a NFRN, such as

$$NFRAAWA_{\omega}\left(\mathfrak{F}_1,\mathfrak{F}_{2,\ldots\ldots,}\mathfrak{F}_n\right)=\oplus_{i=1}^{n}\left(\omega_i\mathfrak{F}_i\right)$$

$$=\left\{\begin{array}{c}\left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega_i\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{d}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega_i\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{i}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega_i\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{b}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right),\\ \left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega_i\left(-\mathbb{LN}\left(1-\overset{\frown}{\mathbb{d}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega_i\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{i}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega_i\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{b}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right)\end{array}\right\}$$

Proof: Selecting the technique of mathematical induction, we have evaluated our result, for this, if $n=1$, thus

$$\omega_1\mathfrak{F}_1=\left\{\begin{array}{c}\left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\omega_1\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\omega_1\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\omega_1\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{b}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right),\\ \left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\omega_1\left(-\mathbb{LN}\left(1-\overset{\frown}{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\omega_1\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\omega_1\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{b}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right)\end{array}\right\},$$

$$\omega_2\mathfrak{F}_2=\left\{\begin{array}{c}\left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\omega_2\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\omega_2\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\omega_2\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{b}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right),\\ \left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\omega_2\left(-\mathbb{LN}\left(1-\overset{\frown}{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\omega_2\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\omega_2\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{b}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right)\end{array}\right\},$$

thus

$\mathrm{NFRAAWA}_{\omega}(\mathfrak{F}_1,\mathfrak{F}_2)=\oplus_{i=1}^{2}(\omega_i\mathfrak{F}_i)=\omega_1\mathfrak{F}_1\oplus\omega_2\mathfrak{F}_2$

$$=\left\{\begin{array}{c}\left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\omega_1\left(-\mathbb{LN}\left(1-\underset{\sim}{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\omega_1\left(-\mathbb{LN}\left(\underset{\sim}{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\omega_1\left(-\mathbb{LN}\left(\underset{\sim}{\mathbb{b}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right),\\ \left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\omega_1\left(-\mathbb{LN}\left(1-\overset{\frown}{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\omega_1\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\omega_1\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{b}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right)\end{array}\right\}\oplus\left\{\begin{array}{c}\left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\omega_2\left(-\mathbb{LN}\left(1-\underset{\sim}{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\omega_2\left(-\mathbb{LN}\left(\underset{\sim}{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\omega_2\left(-\mathbb{LN}\left(\underset{\sim}{\mathbb{b}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right),\\ \left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\omega_2\left(-\mathbb{LN}\left(1-\overset{\frown}{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\omega_2\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\omega_2\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{b}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right)\end{array}\right\}$$

$$=\left\{\begin{array}{c}\left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\omega_1\left(-\mathbb{LN}\left(1-\underset{\sim}{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)+\left(\omega_2\left(-\mathbb{LN}\left(\underset{\sim}{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\omega_1\left(-\mathbb{LN}\left(\underset{\sim}{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)+\left(\omega_2\left(-\mathbb{LN}\left(\underset{\sim}{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \mathbb{E}xp^{-\left(\omega_1\left(-\mathbb{LN}\left(\underset{\sim}{\mathbb{b}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(\omega_2\left(-\mathbb{LN}\left(\underset{\sim}{\mathbb{b}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\end{array}\right),\\ \left(1-\mathbb{E}xp^{-\left(\omega_1\left(-\mathbb{LN}\left(1-\overset{\frown}{\mathbb{d}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(\omega_2\left(-\mathbb{LN}\left(1-\overset{\frown}{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right),\left(\mathbb{E}xp^{-\left(\omega_1\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{i}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(\omega_2\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\right),\\ \mathbb{E}xp^{-\left(\omega_1\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{b}}_{\mathfrak{F}_1}\right)\right)^{\equiv}+\left(\omega_1\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{b}}_{\mathfrak{F}_1}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}\right)}\end{array}\right\}$$

$$=\left\{\begin{array}{c}\left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\sum_{i=1}^{2}\omega_i\left(-\mathbb{LN}\left(1-\underset{\sim}{\mathbb{d}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\sum_{i=1}^{2}\omega_i\left(-\mathbb{LN}\left(\underset{\sim}{\mathbb{i}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{2}\omega_i\left(-\mathbb{LN}\left(\underset{\sim}{\mathbb{b}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right),\\ \left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\sum_{i=1}^{2}\omega_i\left(-\mathbb{LN}\left(1-\overset{\frown}{\mathbb{d}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\sum_{i=1}^{2}\omega_i\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{i}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{2}\omega_i\left(-\mathbb{LN}\left(\overset{\frown}{\mathbb{b}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right)\end{array}\right\}$$

For $n=2$, we correct, further, we assume that for $n=K^{*}$, we also correct, such as

$$NFRAAWA_{\omega}(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,\ldots\ldots,\mathfrak{F}_n)=\oplus_{i=1}^{K^*}(\omega_i\mathfrak{F}_i)$$

$$=\left\{\begin{array}{c}\left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\sum_{i=1}^{K^*}\omega_i\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{d}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\sum_{i=1}^{K^*}\omega_i\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{i}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{K^*}\omega_i\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{b}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right),\\ \left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\sum_{i=1}^{K^*}\omega_i\left(-\mathbb{LN}\left(1-\widehat{\mathbb{d}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\sum_{i=1}^{K^*}\omega_i\left(-\mathbb{LN}\left(\widehat{\mathbb{i}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{K^*}\omega_i\left(-\mathbb{LN}\left(\widehat{\mathbb{b}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right)\end{array}\right\}$$

Then we prove it for $n=K^*+1$, thus

$$NFRAAWA_{\omega}(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,\ldots\ldots,\mathfrak{F}_{K^*+1})=\oplus_{i=1}^{K^*}(\omega_i\mathfrak{F}_i)\oplus(\omega_{K^*+1}\mathfrak{F}_{K^*+1})$$

$$=\left\{\begin{array}{c}\left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\sum_{i=1}^{K^*}\omega_i\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{d}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\sum_{i=1}^{K^*}\omega_i\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{i}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{K^*}\omega_i\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{b}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right),\\ \left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\sum_{i=1}^{K^*}\omega_i\left(-\mathbb{LN}\left(1-\widehat{\mathbb{d}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\sum_{i=1}^{K^*}\omega_i\left(-\mathbb{LN}\left(\widehat{\mathbb{i}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{K^*}\omega_i\left(-\mathbb{LN}\left(\widehat{\mathbb{b}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right)\end{array}\right\}$$

$$\oplus\left\{\begin{array}{c}\left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\omega_{K^*+1}\left(-\mathbb{LN}\left(1-\underset{\smile}{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\omega_{K^*+1}\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\omega_{K^*+1}\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{b}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right),\\ \left(\begin{array}{c}\left(1-\mathbb{E}xp^{-\left(\omega_{K^*+1}\left(-\mathbb{LN}\left(1-\widehat{\mathbb{d}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\left(\mathbb{E}xp^{-\left(\omega_{K^*+1}\left(-\mathbb{LN}\left(\widehat{\mathbb{i}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right),\\ \left(\mathbb{E}xp^{-\left(\omega_{K^*+1}\left(-\mathbb{LN}\left(\widehat{\mathbb{b}}_{\mathfrak{F}_2}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right)\end{array}\right)\end{array}\right\}$$

$$= \left\{ \begin{pmatrix} \left(1 - \mathbb{E}xp^{-\left(\sum_{i=1}^{K^*+1} \omega_i \left(-\mathbb{LN}\left(1 - \underset{\smile}{\mathbb{d}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{K^*+1} \omega_i \left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{i}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{K^*+1} \omega_i \left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{b}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right) \end{pmatrix}, \begin{pmatrix} \left(1 - \mathbb{E}xp^{-\left(\sum_{i=1}^{K^*+1} \omega_i \left(-\mathbb{LN}\left(1 - \widehat{\mathbb{d}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{K^*+1} \omega_i \left(-\mathbb{LN}\left(\widehat{\mathbb{i}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{K^*+1} \omega_i \left(-\mathbb{LN}\left(\widehat{\mathbb{b}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right) \end{pmatrix} \right\}$$

Hence, we proved our target.

Property 1: (Idempotency) If $\mathfrak{F}_i = \mathfrak{F}, i = 1,2, \dots, n$, thus

$$NFRAAWA_{\omega}(\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \dots\dots, \mathfrak{F}_n) = \mathfrak{F}.$$

Proof: We observed that $\mathfrak{F}_i = \left\{\left(\underset{\smile}{\mathbb{d}}_{\mathfrak{F}_i}, \underset{\smile}{\mathbb{i}}_{\mathfrak{F}_i}, \underset{\smile}{\mathbb{b}}_{\mathfrak{F}_i}\right), \left(\widehat{\mathbb{d}}_{\mathfrak{F}_i}, \widehat{\mathbb{i}}_{\mathfrak{F}_i}, \widehat{\mathbb{b}}_{\mathfrak{F}_i}\right)\right\} = \mathfrak{F}, (i = 1,2,3, \dots . n)$, thus

$$NFRAAWA_{\omega}(\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \dots\dots, \mathfrak{F}_{K^*+1}) = \oplus_{i=1}^{n} (\omega_i \mathfrak{F}_i)$$

$$= \left\{ \begin{pmatrix} \left(1 - \mathbb{E}xp^{-\left(\sum_{i=1}^{n} \omega_i \left(-\mathbb{LN}\left(1 - \underset{\smile}{\mathbb{d}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n} \omega_i \left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{i}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n} \omega_i \left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{b}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right) \end{pmatrix}, \begin{pmatrix} \left(1 - \mathbb{E}xp^{-\left(\sum_{i=1}^{n} \omega_i \left(-\mathbb{LN}\left(1 - \widehat{\mathbb{d}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n} \omega_i \left(-\mathbb{LN}\left(\widehat{\mathbb{i}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n} \omega_i \left(-\mathbb{LN}\left(\widehat{\mathbb{b}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right) \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} \left(1 - \mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1 - \underset{\smile}{\mathbb{d}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{i}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\underset{\smile}{\mathbb{b}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right) \end{pmatrix}, \begin{pmatrix} \left(1 - \mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(1 - \widehat{\mathbb{d}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\widehat{\mathbb{i}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(\mathbb{E}xp^{-\left(\left(-\mathbb{LN}\left(\widehat{\mathbb{b}}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right) \end{pmatrix} \right\}$$

$$=\left\{\left(\begin{array}{c}\left(1-\mathbb{E}xp^{\mathbb{LN}\left(1-\underbrace{\mathbb{d}}_{\mathfrak{F}}\right)}\right),\left(\mathbb{E}xp^{\mathbb{LN}\left(\underbrace{\mathbb{i}}_{\mathfrak{F}}\right)}\right),\\ \left(\mathbb{E}xp^{\mathbb{LN}\left(\underbrace{\mathbb{i}}_{\mathfrak{F}}\right)}\right)\end{array}\right),\left(\begin{array}{c}\left(1-\mathbb{E}xp^{\mathbb{LN}\left(1-\underbrace{\mathbb{d}}_{\mathfrak{F}}\right)}\right),\left(\mathbb{E}xp^{\mathbb{LN}\left(\underbrace{\mathbb{i}}_{\mathfrak{F}}\right)}\right),\\ \left(\mathbb{E}xp^{\mathbb{LN}\left(\underbrace{\mathbb{i}}_{\mathfrak{F}}\right)}\right)\end{array}\right)\right\}$$

$$=\left\{\left(\underbrace{\mathbb{d}}_{\mathfrak{F}_i},\underbrace{\mathbb{i}}_{\mathfrak{F}_i},\underbrace{\mathbb{b}}_{\mathfrak{F}_i}\right),\left(\overbrace{\mathbb{d}}_{\mathfrak{F}_i},\overbrace{\mathbb{i}}_{\mathfrak{F}_i},\overbrace{\mathbb{b}}_{\mathfrak{F}_i}\right)\right\}$$

$$=\mathfrak{F}.$$

Property 2: (Boundedness) If $\mathfrak{F}^-=\min(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,\ldots\ldots,\mathfrak{F}_n)$ and $\mathfrak{F}^+=\max(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,\ldots\ldots,\mathfrak{F}_n)$, thus

$$\mathfrak{F}^-\le NFRAAWA_\omega(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,\ldots\ldots,\mathfrak{F}_n)\le\mathfrak{F}^+.$$

Proof: Noticed that $\mathfrak{F}^-=\min(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,\ldots\ldots,\mathfrak{F}_n)=\left\{\left(\underbrace{\mathfrak{F}^-}_{\mathfrak{F}},\underbrace{\mathfrak{F}^-}_{\mathfrak{F}},\underbrace{\mathfrak{F}^-}_{\mathfrak{F}}\right),\left(\overbrace{\mathfrak{F}^-}_{\mathfrak{F}},\overbrace{\mathfrak{F}^-}_{\mathfrak{F}},\overbrace{\mathfrak{F}^-}_{\mathfrak{F}}\right)\right\}$, and $\mathfrak{F}^+=\max(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,\ldots\ldots,\mathfrak{F}_n)=\left\{\left(\underbrace{\mathfrak{F}^+}_{\mathfrak{F}},\underbrace{\mathfrak{F}^+}_{\mathfrak{F}},\underbrace{\mathfrak{F}^+}_{\mathfrak{F}}\right),\left(\overbrace{\mathfrak{F}^+}_{\mathfrak{F}},\overbrace{\mathfrak{F}^+}_{\mathfrak{F}},\overbrace{\mathfrak{F}^+}_{\mathfrak{F}}\right)\right\}$.Now we have $\underbrace{\mathbb{d}^-}_{\mathfrak{F}}=\min_i\left(\underbrace{\mathbb{d}}_{\mathfrak{F}_i}\right),\underbrace{\mathbb{i}^-}_{\mathfrak{F}}=\max_i\left(\underbrace{\mathbb{i}}_{\mathfrak{F}_i}\right),\underbrace{\mathbb{b}^-}_{\mathfrak{F}}=\max_i\left(\underbrace{\xi}_{\mathfrak{F}_i}\right),\underbrace{\mathbb{d}^+}_{\mathfrak{F}}=\max_i\left(\underbrace{\mu}_{\mathfrak{F}_i}\right),\underbrace{\mathbb{i}^+}_{\mathfrak{F}}=\min_i\left(\underbrace{\eta}_{\mathfrak{F}_i}\right),\underbrace{\mathbb{b}^+}_{\mathfrak{F}}=\min_i\left(\underbrace{\xi}_{\mathfrak{F}_i}\right)$ and $\overbrace{\mathbb{d}^-}_{\mathfrak{F}}=\min_i\left(\overbrace{\mathbb{d}}_{\mathfrak{F}_i}\right),\overbrace{\mathbb{i}^-}_{\mathfrak{F}}=\max_i\left(\overbrace{\mathbb{i}}_{\mathfrak{F}_i}\right),\overbrace{\mathbb{b}^-}_{\mathfrak{F}}=\max_i\left(\overbrace{\mathbb{b}}_{\mathfrak{F}_i}\right),\overbrace{\mathbb{d}^+}_{\mathfrak{F}}=\max_i\left(\overbrace{\mathbb{d}}_{\mathfrak{F}_i}\right),\overbrace{\mathbb{i}^+}_{\mathfrak{F}}=\min_i\left(\overbrace{\mathbb{i}}_{\mathfrak{F}_i}\right),\overbrace{\mathbb{b}^+}_{\mathfrak{F}}=\min_i\left(\overbrace{\mathbb{b}}_{\mathfrak{F}_i}\right)$, thus

$$1-\mathbb{E}xp^{-\left(\sum_{i=1}^n\omega_i\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{d}^-}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\le 1-\mathbb{E}xp^{-\left(\sum_{i=1}^n\omega_i\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{d}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\le 1-\mathbb{E}xp^{-\left(\sum_{i=1}^n\omega_i\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{d}^+}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}$$

$$1-\mathbb{E}xp^{-\left(\sum_{i=1}^n\omega_i\left(-\mathbb{LN}\left(1-\overbrace{\mathbb{d}^-}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\le 1-\mathbb{E}xp^{-\left(\sum_{i=1}^n\omega_i\left(-\mathbb{LN}\left(1-\overbrace{\mathbb{d}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\le 1-\mathbb{E}xp^{-\left(\sum_{i=1}^n\omega_i\left(-\mathbb{LN}\left(1-\overbrace{\mathbb{d}^+}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}$$

Furthermore,

$$\mathbb{E}xp^{-\left(\sum_{i=1}^n\omega_i\left(-\mathbb{LN}\left(\underbrace{\mathbb{i}^+}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\le\mathbb{E}xp^{-\left(\sum_{i=1}^n\omega_i\left(-\mathbb{LN}\left(\underbrace{\mathbb{i}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\le\mathbb{E}xp^{-\left(\sum_{i=1}^n\omega_i\left(-\mathbb{LN}\left(\underbrace{\mathbb{i}^-}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}$$

$$\mathbb{E}xp^{-\left(\sum_{i=1}^n\omega_i\left(-\mathbb{LN}\left(\overbrace{\mathbb{i}^+}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\le\mathbb{E}xp^{-\left(\sum_{i=1}^n\omega_i\left(-\mathbb{LN}\left(\overbrace{\mathbb{i}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\le\mathbb{E}xp^{-\left(\sum_{i=1}^n\omega_i\left(-\mathbb{LN}\left(\overbrace{\mathbb{i}^-}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}$$

and

$$\mathbb{E}xp^{-\left(\sum_{i=1}^n\omega_i\left(-\mathbb{LN}\left(\underbrace{\mathbb{b}^+}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\le\mathbb{E}xp^{-\left(\sum_{i=1}^n\omega_i\left(-\mathbb{LN}\left(\underbrace{\mathbb{b}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\le\mathbb{E}xp^{-\left(\sum_{i=1}^n\omega_i\left(-\mathbb{LN}\left(\underbrace{\mathbb{b}^-}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}$$

$$\mathbb{E}xp^{-\left(\sum_{i=1}^n\omega_i\left(-\mathbb{LN}\left(\overbrace{\mathbb{b}^+}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\le\mathbb{E}xp^{-\left(\sum_{i=1}^n\omega_i\left(-\mathbb{LN}\left(\overbrace{\mathbb{b}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\le\mathbb{E}xp^{-\left(\sum_{i=1}^n\omega_i\left(-\mathbb{LN}\left(\overbrace{\mathbb{b}^-}_{\mathfrak{F}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}$$

Then,

$$\mathfrak{F}^-\le NFRAAWA_\omega(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,\ldots\ldots,\mathfrak{F}_n)\le\mathfrak{F}^+$$

Property 3: (Monotonicity) If $\mathfrak{F}_i\le\mathfrak{F}'_i, i=1,2,\ldots,n$, thus

$$NFRAAWA_\omega(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,\ldots\ldots,\mathfrak{F}_n)\le NFRAAWA_\omega(\mathfrak{F}'_1,\mathfrak{F}'_2,\mathfrak{F}'_3,\ldots\ldots,\mathfrak{F}'_n).$$

Proof: Assume that $\mathfrak{F}_i\le\mathfrak{F}'_i$, thus

$$\mathfrak{F}_i\le\mathfrak{F}'_i\Leftrightarrow\underbrace{\mathbb{d}}_{\mathfrak{F}_i}\le\underbrace{\mathbb{d}'}_{\mathfrak{F}_i},\underbrace{\mathbb{i}}_{\mathfrak{F}_i}\ge\underbrace{\mathbb{i}'}_{\mathfrak{F}_i},\underbrace{\mathbb{b}}_{\mathfrak{F}_i}\ge\underbrace{\mathbb{b}'}_{\mathfrak{F}_i}$$

Furthermore,

$$\underbrace{\mathbb{d}}_{\mathfrak{F}_i}\le\underbrace{\mathbb{d}'}_{\mathfrak{F}_i}\Rightarrow 1-\underbrace{\mathbb{d}}_{\mathfrak{F}_i}\ge 1-\underbrace{\mathbb{d}'}_{\mathfrak{F}_i}$$

$$\Rightarrow-\mathbb{LN}\left(1-\underbrace{\mathbb{d}}_{\mathfrak{F}_i}\right)\le-\mathbb{LN}\left(1-\underbrace{\mathbb{d}'}_{\mathfrak{F}_i}\right)$$

$$\Rightarrow \left(-\mathbb{LN}\left(1-\underbrace{\mathbb{d}}_{\mathfrak{F}_i}\right)\right)^{\equiv} \leq \left(-\mathbb{LN}\left(1-\underbrace{\mathbb{d}'}_{\mathfrak{F}_i}\right)\right)^{\equiv}$$

$$\Rightarrow \left(\sum_{i=1}^{n} \omega_i \left(-\mathbb{LN}\left(1-\underbrace{\mathbb{d}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}} \leq \left(\sum_{i=1}^{n} \omega_i \left(-\mathbb{LN}\left(1-\underbrace{\mathbb{d}'}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}$$

$$\Rightarrow -\left(\sum_{i=1}^{n} \omega_i \left(-\mathbb{LN}\left(1-\underbrace{\mathbb{d}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}} \geq -\left(\sum_{i=1}^{n} \omega_i \left(-\mathbb{LN}\left(1-\underbrace{\mathbb{d}'}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}$$

$$\Rightarrow 1-\mathbb{E}xp^{-\left(\sum_{i=1}^{n} \omega_i \left(-\mathbb{LN}\left(1-\underbrace{\mathbb{d}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \leq 1-\mathbb{E}xp^{-\left(\sum_{i=1}^{n} \omega_i \left(-\mathbb{LN}\left(1-\underbrace{\mathbb{d}'}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}$$

And

$$\underbrace{\mathbb{i}}_{\mathfrak{F}_i} \geq \underbrace{\mathbb{i}'}_{\mathfrak{F}_i}$$

$$\Rightarrow -log\left(\underbrace{\mathbb{i}}_{\mathfrak{F}_i}\right) \leq -log\left(\underbrace{\mathbb{i}'}_{\mathfrak{F}_i}\right)$$

$$\Rightarrow \left(-log\left(\underbrace{\mathbb{i}}_{\mathfrak{F}_i}\right)\right)^{\equiv} \leq \left(-log\left(\underbrace{\mathbb{i}'}_{\mathfrak{F}_i}\right)\right)^{\equiv}$$

$$\Rightarrow \left(\sum_{i=1}^{n} \omega_i \left(-\mathbb{LN}\left(\underbrace{\mathbb{i}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}} \leq \left(\sum_{i=1}^{n} \omega_i \left(-\mathbb{LN}\left(\underbrace{\mathbb{i}'}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}$$

$$\Rightarrow -\left(\sum_{i=1}^{n} \omega_i \left(-\mathbb{LN}\left(\underbrace{\mathbb{i}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}} \geq -\left(\sum_{i=1}^{n} \omega_i \left(-\mathbb{LN}\left(\underbrace{\mathbb{i}'}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}$$

$$\Rightarrow \mathbb{E}xp^{-\left(\sum_{i=1}^{n} \omega_i \left(-\mathbb{LN}\left(\underbrace{\mathbb{i}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \geq \mathbb{E}xp^{-\left(\sum_{i=1}^{n} \omega_i \left(-\mathbb{LN}\left(\underbrace{\mathbb{i}'}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}$$

And similarly for falsity information, such as

$$\Rightarrow \mathbb{E}xp^{-\left(\sum_{i=1}^{n} \omega_i \left(-\mathbb{LN}\left(\underbrace{\mathbb{b}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \geq \mathbb{E}xp^{-\left(\sum_{i=1}^{n} \omega_i \left(-\mathbb{LN}\left(\underbrace{\mathbb{b}'}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}$$

Similarly, for upper approximation, we have to prove truth, falsity, and, abstinence by using the above information, we can easily derive our required information, such as

$$NFRAAWA_{\omega}(\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \ldots\ldots, \mathfrak{F}_n) \leq NFRAAWA_{\omega}(\mathfrak{F}'_1, \mathfrak{F}'_2, \mathfrak{F}'_3, \ldots\ldots, \mathfrak{F}'_n).$$

Definition 12: The NFRAAOWA operator is explained and defined below, such as

$$NFRAAOWA : (\Xi^l) \rightarrow \Xi^l$$

$$NFRAAOWA(\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \ldots\ldots, \mathfrak{F}_n) = \oplus_{i=1}^{n}\left(\omega^*{}_i \mathfrak{F}_{O(i)}\right) = \omega^*{}_1 \mathfrak{F}_{O(1)} \oplus \omega^*{}_2 \mathfrak{F}_{O(2)} \oplus \ldots \oplus \omega^*{}_n \mathfrak{F}_{O(n)}$$

where $\mathbb{d}_{O(n-1)} \geq \mathbb{d}_{O(n)}$.

Theorem 3: To consider the data in the above information, we derive that our obtained result will be also a NFRN, such as

$$NFRAAOWA_{\omega}(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,\ldots\ldots,\mathfrak{F}_n)$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \left(1-\mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega^*{}_i\left(-\mathbb{LN}\left(1-\underset{\sim}{\mathbb{d}}_{\mathfrak{F}_{O(i)}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega^*{}_i\left(-\mathbb{LN}\left(\underset{\sim}{\mathbb{i}}_{\mathfrak{F}_{O(i)}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega_i\left(-\mathbb{LN}\left(\underset{\sim}{\mathbb{b}}_{\mathfrak{F}_{O(i)}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right) \end{array} \right), \\ \left(\begin{array}{c} \left(1-\mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega^*{}_i\left(-\mathbb{LN}\left(1-\widehat{\mathbb{d}}_{\mathfrak{F}_{O(i)}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega^*{}_i\left(-\mathbb{LN}\left(\widehat{\mathbb{i}}_{\mathfrak{F}_{O(i)}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega^*{}_i\left(-\mathbb{LN}\left(\widehat{\mathbb{b}}_{\mathfrak{F}_{O(i)}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right) \end{array} \right) \end{array} \right\}$$

Property 4: (Idempotency) If $\mathfrak{F}_i = \mathfrak{F}$, thus

$$NFRAAOWA_{\omega^*{}_i}(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,\ldots\ldots,\mathfrak{F}_n) = A.$$

Property 5: (Boundedness) Consider $\mathfrak{F}^- = \min(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,\ldots\ldots,\mathfrak{F}_n)$ and $\mathfrak{F}^+ = \max(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,\ldots\ldots,\mathfrak{F}_n)$, thus

$$\mathfrak{F}^- \le NFRAAOWA_{\omega}(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,\ldots\ldots,\mathfrak{F}_n) \le \mathfrak{F}^+.$$

Property 6: (Monotonicity) If $\mathfrak{F}_i \le \mathfrak{F}'{}_i$, thus

$$NFRAAOWA_{\omega^*{}_i}(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,\ldots\ldots,\mathfrak{F}_n) \le NFRAAOWA_{\omega^*{}_i}(\mathfrak{F}'_1,\mathfrak{F}'_2,\mathfrak{F}'_3,\ldots\ldots,\mathfrak{F}'_n).$$

Definition 13: The NFRAAHA operator is invented and defined, such as

$$NFRAAHA : (\Xi^l) \to \Xi^l$$

$$NFRAAHA_{\omega,\omega^*}(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,\ldots\ldots,\mathfrak{F}_n) = \oplus_{i=1}^{n}\left(\omega^*{}_i\mathfrak{F}_{O(i)}\right) = \omega^*{}_1\mathfrak{F}_{O(1)}\oplus\omega^*{}_2\mathfrak{F}_{O(2)}\oplus\ldots\oplus\omega^*{}_n\mathfrak{F}_{O(n)}$$

Noticed that $\mathfrak{F}_i = n\omega_i\mathfrak{F}_i (i = 1,2,3,\ldots,n)$ with $\mathfrak{F}_{O(n-1)} \ge \mathfrak{F}_{O(n)}$.

Theorem 4: To consider the data in the above information, we derive that our obtained result will be also a NFRN, such as

$NFRAAOWA_{\omega,\omega^*}(\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \ldots\ldots, \mathfrak{F}_n)$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \left(1 - \mathbb{E}xp^{-\left(\sum_{i=1}^{n} \omega^*_i \left(-\mathbb{LN}\left(1 - \underbrace{\mathbb{d}}_{\mathfrak{F}_{O(i)}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n} \omega^*_i \left(-\mathbb{LN}\left(\underbrace{\mathbb{i}}_{\mathfrak{F}_{O(i)}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n} \omega_i \left(-\mathbb{LN}\left(\underbrace{\mathbb{b}}_{\mathfrak{F}_{O(i)}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right) \end{array} \right), \\ \left(\begin{array}{c} \left(1 - \mathbb{E}xp^{-\left(\sum_{i=1}^{n} \omega^*_i \left(-\mathbb{LN}\left(1 - \overbrace{\mathbb{d}}_{\mathfrak{F}_{O(i)}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n} \omega^*_i \left(-\mathbb{LN}\left(\overbrace{\mathbb{i}}_{\mathfrak{F}_{O(i)}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right), \\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n} \omega^*_i \left(-\mathbb{LN}\left(\overbrace{\mathbb{b}}_{\mathfrak{F}_{O(i)}}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}} \right) \end{array} \right) \end{array} \right\}$$

Property 7: (Idempotency) If $\mathfrak{F}_i = \mathfrak{F}$, thus

$$NFRAAHA_{\omega^*_i}(\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \ldots\ldots, \mathfrak{F}_n) = A.$$

Property 8: (Boundedness) Consider $\mathfrak{F}^- = \min(\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \ldots\ldots, \mathfrak{F}_n)$ and $\mathfrak{F}^+ = \max(\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \ldots\ldots, \mathfrak{F}_n)$, thus

$$\mathfrak{F}^- \leq NFRAAHA_{\omega}(\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \ldots\ldots, \mathfrak{F}_n) \leq \mathfrak{F}^+.$$

Property 9: (Monotonicity) If $\mathfrak{F}_i \leq \mathfrak{F}'_i$, thus

$$NFRAAHA_{\omega^*_i}(\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \ldots\ldots, \mathfrak{F}_n) \leq NFRAAHA_{\omega^*_i}(\mathfrak{F}'_1, \mathfrak{F}'_2, \mathfrak{F}'_3, \ldots\ldots, \mathfrak{F}'_n).$$

5 | MADM TECHNIQUE BASED ON PROPOSED OPERATORS

In this section, we state the supremacy and validity of the proposed operators by utilizing or implementing the proposed techniques in the environment of the MADM technique to enhance the worth of the stated methods.

For this, we select m alternatives $\mathfrak{J}^* = \{\mathfrak{J}^*_1, \mathfrak{J}^*_2, \ldots, \mathfrak{J}^*_m\}$ and n attributes $P = \{P_1, P_2, \ldots, P_n\}$, where these all attributes are stated for each alternative with weight vector such as: $\omega = (\omega_1, \omega_2, \ldots\ldots, \omega_n)^{\mathbb{T}}$ with $\sum_{i=1}^{n} \omega_i = 1$. For the above data, we compute the decision matrix by including their values in the shape of NFRNs, such as $\mathfrak{F}_{vi} = \left\{ \left(\underbrace{\mathbb{d}}_{\mathfrak{F}_i}, \underbrace{\mathbb{i}}_{\mathfrak{F}_i}, \underbrace{\mathbb{b}}_{\mathfrak{F}_i} \right), \left(\overbrace{\mathbb{d}}_{\mathfrak{F}_i}, \overbrace{\mathbb{i}}_{\mathfrak{F}_i}, \overbrace{\mathbb{b}}_{\mathfrak{F}_i} \right) \right\}$, thus

$$\mathcal{R} = (\mathfrak{F}_{vi})_{(m\times n)}$$

$$= \begin{bmatrix} \left\{\begin{matrix}\left(\underbrace{\mathbb{d}}_{\mathfrak{F}_{11}}, \underbrace{\mathbb{i}}_{\mathfrak{F}_{11}}, \underbrace{\mathbb{b}}_{\mathfrak{F}_{11}}\right), \\ \left(\overbrace{\mathbb{d}}_{\mathfrak{F}_{11}}, \overbrace{\mathbb{i}}_{\mathfrak{F}_{11}}, \overbrace{\mathbb{b}}_{\mathfrak{F}_{11}}\right)\end{matrix}\right\} & \left\{\left(\underbrace{\mathbb{d}}_{\mathfrak{F}_{12}}, \underbrace{\mathbb{i}}_{\mathfrak{F}_{12}}, \underbrace{\mathbb{b}}_{\mathfrak{F}_{12}}\right), \left(\overbrace{\mathbb{d}}_{\mathfrak{F}_{12}}, \overbrace{\mathbb{i}}_{\mathfrak{F}_{12}}, \overbrace{\mathbb{b}}_{\mathfrak{F}_{12}}\right)\right\} & \cdots \left\{\begin{matrix}\left(\underbrace{\mathbb{d}}_{\mathfrak{F}_{1n}}, \underbrace{\mathbb{i}}_{\mathfrak{F}_{1n}}, \underbrace{\mathbb{b}}_{\mathfrak{F}_{1n}}\right), \\ \left(\overbrace{\mathbb{d}}_{\mathfrak{F}_{1n}}, \overbrace{\mathbb{i}}_{\mathfrak{F}_{1n}}, \overbrace{\mathbb{b}}_{\mathfrak{F}_{1n}}\right)\end{matrix}\right\} \\ \left\{\begin{matrix}\left(\underbrace{\mathbb{d}}_{\mathfrak{F}_{21}}, \underbrace{\mathbb{i}}_{\mathfrak{F}_{21}}, \underbrace{\mathbb{b}}_{\mathfrak{F}_{21}}\right), \\ \left(\overbrace{\mathbb{d}}_{\mathfrak{F}_{21}}, \overbrace{\mathbb{i}}_{\mathfrak{F}_{21}}, \overbrace{\mathbb{b}}_{\mathfrak{F}_{21}}\right)\end{matrix}\right\} & \left\{\left(\underbrace{\mathbb{d}}_{\mathfrak{F}_{22}}, \underbrace{\mathbb{i}}_{\mathfrak{F}_{22}}, \underbrace{\mathbb{b}}_{\mathfrak{F}_{22}}\right), \left(\overbrace{\mathbb{d}}_{\mathfrak{F}_{22}}, \overbrace{\mathbb{i}}_{\mathfrak{F}_{22}}, \overbrace{\mathbb{b}}_{\mathfrak{F}_{22}}\right)\right\} & \cdots \left\{\begin{matrix}\left(\underbrace{\mathbb{d}}_{\mathfrak{F}_{2n}}, \underbrace{\mathbb{i}}_{\mathfrak{F}_{2n}}, \underbrace{\mathbb{b}}_{\mathfrak{F}_{2n}}\right), \\ \left(\overbrace{\mathbb{d}}_{\mathfrak{F}_{2n}}, \overbrace{\mathbb{i}}_{\mathfrak{F}_{2n}}, \overbrace{\mathbb{b}}_{\mathfrak{F}_{2n}}\right)\end{matrix}\right\} \\ & \vdots & \\ \left\{\begin{matrix}\left(\underbrace{\mathbb{d}}_{\mathfrak{F}_{m1}}, \underbrace{\mathbb{i}}_{\mathfrak{F}_{m1}}, \underbrace{\mathbb{b}}_{\mathfrak{F}_{m1}}\right). \\ \left(\overbrace{\mathbb{d}}_{\mathfrak{F}_{m1}}, \overbrace{\mathbb{i}}_{\mathfrak{F}_{m1}}, \overbrace{\mathbb{b}}_{m1}\right)\end{matrix}\right\} & \left\{\left(\underbrace{\mathbb{d}}_{m2}, \underbrace{\mathbb{i}}_{\mathfrak{F}_{m2}}, \underbrace{\mathbb{b}}_{\mathfrak{F}_{m2}}\right) \left(\overbrace{\mathbb{d}}_{\mathfrak{F}_{m2}}, \overbrace{\mathbb{i}}_{\mathfrak{F}_{m2}}, \overbrace{\mathbb{b}}_{\mathfrak{F}_{m2}}\right)\right\} & \cdots \left\{\begin{matrix}\left(\underbrace{\mathbb{d}}_{\mathfrak{F}_{mn}}, \underbrace{\mathbb{i}}_{\mathfrak{F}_{mn}}, \underbrace{\mathbb{b}}_{\mathfrak{F}_{mn}}\right), \\ \left(\underbrace{\mathbb{d}}_{\mathfrak{F}_{mn}}, \underbrace{\mathbb{i}}_{\mathfrak{F}_{mn}}, \underbrace{\mathbb{b}}_{\mathfrak{F}_{mn}}\right)\end{matrix}\right\} \end{bmatrix}$$

Noticed that $\underbrace{\mathbb{d}_{\delta}}_{\mathcal{M}}(\varkappa) = \wedge_{z\in p}[\mathbb{d}_{\delta}(\varkappa, z)\wedge \mathbb{d}_{\mathcal{M}}(z)]$, $\underbrace{\mathbb{i}_{\delta}}_{\mathcal{M}}(\varkappa) = \vee_{z\in p}[\mathbb{i}_{\delta}(\varkappa, z)\vee \mathbb{i}_{\mathcal{M}}(z)]$ and $\underbrace{\mathbb{b}_{\delta}}_{\mathcal{M}}(\varkappa) = \vee_{z\in p}[\mathbb{b}_{\delta}(\varkappa, z)\vee \mathbb{b}_{\mathcal{M}}(z)]$ and $\overbrace{\mathbb{d}_{\delta}}_{\mathcal{M}}(\varkappa) = \vee_{z\in p}[\mathbb{d}_{\delta}(\varkappa, z)\vee \mathbb{d}_{\mathcal{M}}(z)]$, $\overbrace{\mathbb{i}_{\delta}}_{\mathcal{M}}(\varkappa) = \wedge_{z\in p}[\mathbb{i}_{\delta}(\varkappa, z)\wedge \mathbb{i}_{\mathcal{M}}(z)]$ and $\overbrace{\mathbb{b}_{\delta}}_{\mathcal{M}}(\varkappa) = \wedge_{z\in p}[\mathbb{b}_{\delta}(\varkappa, z)\wedge \mathbb{b}_{\mathcal{M}}(z)]$ with $0 \le \underbrace{\mathbb{d}_{\delta}}_{\mathcal{M}}(\varkappa) + \underbrace{\mathbb{i}_{\delta}}_{\mathcal{M}}(\varkappa) + \underbrace{\mathbb{b}_{\delta}}_{\mathcal{M}}(\varkappa) \le 3, 0 \le \overbrace{\mathbb{d}_{\delta}}_{\mathcal{M}}(\varkappa) + \overbrace{\mathbb{i}_{\delta}}_{\mathcal{M}}(\varkappa) + \overbrace{\mathbb{b}_{\delta}}_{\mathcal{M}}(\varkappa) \le 3$. Further, $\underbrace{\delta}(\mathcal{M})$ and $\overbrace{\delta}(\mathcal{M})$ represents the NFRs with a function $\underbrace{\delta}(\mathcal{M}), \overbrace{\delta}(\mathcal{M}): NFS_{(p)\to}NFS_{(p)}$, called LR and UR approximation operators. Finally, $\delta(\mathcal{M}) = \left(\underbrace{\delta}(\mathcal{M}), \overbrace{\delta}(\mathcal{M})\right) = \left\{\varkappa: \left(\underbrace{\mathbb{d}_{\delta}}_{\mathcal{M}}(\varkappa), \underbrace{\mathbb{i}_{\delta}}_{\mathcal{M}}(\varkappa), \underbrace{\mathbb{b}_{\delta}}_{\mathcal{M}}(\varkappa)\right), \left(\overbrace{\mathbb{d}_{\delta}}_{\mathcal{M}}(\varkappa), \overbrace{\mathbb{i}_{\delta}}_{\mathcal{M}}(\varkappa), \overbrace{\mathbb{b}_{\delta}}_{\mathcal{M}}(\varkappa)\right) | \varkappa \in p\right\}$ represents the NFRS with $\mathbb{d}(\mathcal{M}) = \left(\underbrace{\delta}(\mathcal{M}), \overbrace{\delta}(\mathcal{M})\right) = \left\{\varkappa: \left(\underbrace{\mathbb{d}_{\delta}}_{\mathcal{M}}(\varkappa), \underbrace{\mathbb{i}_{\delta}}_{\mathcal{M}}(\varkappa), \underbrace{\mathbb{b}_{\delta}}_{\mathcal{M}}(\varkappa)\right), \left(\overbrace{\mathbb{d}_{\delta}}_{\mathcal{M}}(\varkappa), \overbrace{\mathbb{i}_{\delta}}_{\mathcal{M}}(\varkappa), \overbrace{\mathbb{b}_{\delta}}_{\mathcal{M}}(\varkappa)\right) \varkappa \in p\right\}$, with a simple shape: $\delta(\mathcal{M}) = \left\{\left(\underbrace{\mathbb{d}}_{\mathfrak{F}}, \underbrace{\mathbb{i}}_{\mathfrak{F}}, \underbrace{\mathbb{b}}_{\mathfrak{F}}\right), \left(\overbrace{\mathbb{d}}_{\mathfrak{F}}, \overbrace{\mathbb{i}}_{\mathfrak{F}}, \overbrace{\mathbb{b}}_{\mathfrak{F}}\right)\right\}$, also called NFRNs. Thus, we demonstrate the procedure MADM, such as

Step 1: After constructing the matrix, we aim to normalize the matrix by using the below information, such as $\mathcal{R} = \begin{cases}(\mathfrak{F}_{vi})^{c}; for\ cost \\ \mathfrak{F}_{vi}; for\ benifit\end{cases}$, where $(\mathfrak{F}_{vi})^{c} = \left\{\begin{matrix}\left(\underbrace{\mathbb{b}}_{\mathfrak{F}_{i}}, \underbrace{\mathbb{i}}_{\mathfrak{F}_{i}}, \underbrace{\mathbb{d}}_{\mathfrak{F}_{i}}\right), \\ \left(\overbrace{\mathbb{b}}_{\mathfrak{F}_{i}}, \overbrace{\mathbb{i}}_{\mathfrak{F}_{i}}, \overbrace{\mathbb{d}}_{\mathfrak{F}_{i}}\right)\end{matrix}\right\}$, represents the complement of $\mathfrak{F}_{vi} = \left\{\left(\underbrace{\mathbb{d}}_{\mathfrak{F}_{i}}, \underbrace{\mathbb{i}}_{\mathfrak{F}_{i}}, \underbrace{\mathbb{b}}_{\mathfrak{F}_{i}}\right), \left(\overbrace{\mathbb{d}}_{\mathfrak{F}_{i}}, \overbrace{\mathbb{i}}_{\mathfrak{F}_{i}}, \overbrace{\mathbb{b}}_{\mathfrak{F}_{i}}\right)\right\}$.

Step 2: Aggregate the information in the normalized matrix by using the theory of NFRAAWA operator, such as

$$\mathfrak{F}_v = NFRAAWA_\omega(\mathfrak{F}_{v1}, \mathfrak{F}_{v2}, \ldots\ldots, \mathfrak{F}_{vn}) = \oplus_{i=1}^{n}(\omega_i \mathfrak{F}_{vi})$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \left(1 - \mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega_i\left(-\mathbb{LN}\left(1-\underbrace{\mathbb{d}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega_i\left(-\mathbb{LN}\left(\underbrace{\mathbb{i}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega_i\left(-\mathbb{LN}\left(\underbrace{\mathbb{b}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right) \end{array} \right), \\ \left(\begin{array}{c} \left(1 - \mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega_i\left(-\mathbb{LN}\left(1-\overbrace{\mathbb{d}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega_i\left(-\mathbb{LN}\left(\overbrace{\mathbb{i}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right), \\ \left(\mathbb{E}xp^{-\left(\sum_{i=1}^{n}\omega_i\left(-\mathbb{LN}\left(\overbrace{\mathbb{b}}_{\mathfrak{F}_i}\right)\right)^{\equiv}\right)^{\frac{1}{\equiv}}}\right) \end{array} \right) \end{array} \right\}$$

Step 3: Evaluate the score values of the above-aggregated information, such as

$$Sc(\mathfrak{F}) = \frac{1}{3}\left(\underbrace{\mathbb{d}}_{\mathfrak{F}} + \overbrace{\mathbb{d}}_{\mathfrak{F}} - \underbrace{\mathbb{i}}_{\mathfrak{F}} - \overbrace{\mathbb{i}}_{\mathfrak{F}} - \underbrace{\mathbb{b}}_{\mathfrak{F}} - \overbrace{\mathbb{b}}_{\mathfrak{F}}\right), Sc(\mathfrak{F}) \in [0,1]$$

If the idea of score value has failed, then we will use the accuracy value, such as

$$Ac(\mathfrak{F}) = \frac{1}{3}\left(\underbrace{\mathbb{d}}_{\mathfrak{F}} + \overbrace{\mathbb{d}}_{\mathfrak{F}} + \underbrace{\mathbb{i}}_{\mathfrak{F}} + \overbrace{\mathbb{i}}_{\mathfrak{F}} + \underbrace{\mathbb{b}}_{\mathfrak{F}} + \overbrace{\mathbb{b}}_{\mathfrak{F}}\right), Ac(\mathfrak{F}) \in [0,1].$$

Step 4: Under consideration of the above information, we rank all the alternatives and find the best one.

To simplify or evaluate some practical examples based on the above information, we consider the problems from chemistry based on the X-ray analysis of crystal𝕃ℕraphy or crystal structure based on the invented operators to enhance the worth of the above information.

5.1 | X-RAY ANALYSIS OF CRYSTAL STRUCTURE BASED ON NFR INFORMATION

A potent method for figuring out the three-dimensional atomic structure of a crystalline substance, whether a chemical compound or a bio𝕃ℕical macromolecule (like proteins and nucleic acids), is X-ray crystal𝕃ℕraphy. To produce an intricate structural model depends on the interaction between X-rays and the electron density inside the crystal. In this application, we find the most dominant part of the X-ray analysis for crystal based on the proposed techniques, such as

1. Crystal Preparation "$\mathfrak{I}^*_1$".
2. X-ray Diffraction """$\mathfrak{I}^*_2$""".
3. Fourier Transformation """$\mathfrak{I}^*_3$""".
4. Refinement """$\mathfrak{I}^*_4$""".
5. Structure Information """$\mathfrak{I}^*_5$""

To choose the best one among the above five, we have the following criteria, such as growth analysis, political impact, social impact, environmental impact, and market levels with the following weighted vectors as $(0.2,0.2,0.2,0.2,0.2)^T$. Thus, we demonstrate the procedure MADM, such as

Step 1: After constructing the matrix in the shape of Table 1, we aim to normalize the matrix by using the below information, such as $\mathcal{R} = \begin{cases} (\mathfrak{F}_{vi})^c; for\ cost \\ \mathfrak{F}_{vi}; for benifit \end{cases}$, where $(\mathfrak{F}_{vi})^c = \begin{Bmatrix} \left(\underbrace{\mathbb{b}}_{\mathfrak{F}_i}, \underbrace{\mathbb{i}}_{\mathfrak{F}_i}, \underbrace{\mathbb{d}}_{\mathfrak{F}_i}\right), \\ \left(\overbrace{\mathbb{b}}_{\mathfrak{F}_i}, \overbrace{\mathbb{i}}_{\mathfrak{F}_i}, \overbrace{\mathbb{d}}_{\mathfrak{F}_i}\right) \end{Bmatrix}$, represents the complement of $\mathfrak{F}_{vi} = \left\{\left(\underbrace{\mathbb{d}}_{\mathfrak{F}_i}, \underbrace{\mathbb{i}}_{\mathfrak{F}_i}, \underbrace{\mathbb{b}}_{\mathfrak{F}_i}\right), \left(\overbrace{\mathbb{d}}_{\mathfrak{F}_i}, \overbrace{\mathbb{i}}_{\mathfrak{F}_i}, \overbrace{\mathbb{b}}_{\mathfrak{F}_i}\right)\right\}$. Anyhow, the data in Table 1 is not required to be evaluated.

Table 1. NFR decision information.

	P_1	P_2	P_3	P_4	P_5
$\mathfrak{I}^*{}_1$	$\begin{pmatrix}(0.6,0.4,0.7),\\(0.5,0.8,0.7)\end{pmatrix}$	$\begin{pmatrix}(0.61,0.4,0.71),\\(0.51,0.81,0.71)\end{pmatrix}$	$\begin{pmatrix}(0.62,0.42,0.72),\\(0.52,0.82,0.72)\end{pmatrix}$	$\begin{pmatrix}(0.63,0.43,0.73),\\(0.53,0.83,0.73)\end{pmatrix}$	$\begin{pmatrix}(0.64,0.44,0.74),\\(0.54,0.84,0.74)\end{pmatrix}$
$\mathfrak{I}^*{}_2$	$\begin{pmatrix}(0.7,0.7,0.5),\\(0.9,0.5,0.8)\end{pmatrix}$	$\begin{pmatrix}(0.71,0.71,0.51),\\(0.91,0.51,0.81)\end{pmatrix}$	$\begin{pmatrix}(0.72,0.72,0.52),\\(0.92,0.52,0.82)\end{pmatrix}$	$\begin{pmatrix}(0.73,0.73,0.5),\\(0.93,0.53,0.8)\end{pmatrix}$	$\begin{pmatrix}(0.74,0.74,0.54),\\(0.94,0.54,0.84)\end{pmatrix}$
$\mathfrak{I}^*{}_3$	$\begin{pmatrix}(0.8,0.6,0.9),\\(0.7,0.7,0.5)\end{pmatrix}$	$\begin{pmatrix}(0.81,0.61,0.91),\\(0.71,0.71,0.51)\end{pmatrix}$	$\begin{pmatrix}(0.82,0.62,0.92),\\(0.72,0.72,0.52)\end{pmatrix}$	$\begin{pmatrix}(0.83,0.63,0.93),\\(0.73,0.73,0.53)\end{pmatrix}$	$\begin{pmatrix}(0.84,0.64,0.94),\\(0.74,0.74,0.54)\end{pmatrix}$
$\mathfrak{I}^*{}_4$	$\begin{pmatrix}(0.6,0.9,0.6),\\(0.6,0.5,0.7)\end{pmatrix}$	$\begin{pmatrix}(0.61,0.91,0.61),\\(0.61,0.51,0.71)\end{pmatrix}$	$\begin{pmatrix}(0.62,0.92,0.62),\\(0.62,0.52,0.72)\end{pmatrix}$	$\begin{pmatrix}(0.63,0.93,0.63),\\(0.63,0.53,0.73)\end{pmatrix}$	$\begin{pmatrix}(0.64,0.94,0.64),\\(0.64,0.54,0.74)\end{pmatrix}$
$\mathfrak{I}^*{}_5$	$\begin{pmatrix}(0.8,0.5,0.6),\\(0.8,0.7,0.6)\end{pmatrix}$	$\begin{pmatrix}(0.81,0.51,0.61),\\(0.81,0.71,0.61)\end{pmatrix}$	$\begin{pmatrix}(0.82,0.52,0.62),\\(0.82,0.72,0.62)\end{pmatrix}$	$\begin{pmatrix}(0.83,0.53,0.63),\\(0.83,0.73,0.63)\end{pmatrix}$	$\begin{pmatrix}(0.84,0.54,0.64),\\(0.84,0.74,0.64)\end{pmatrix}$

Step 2: Aggregate the information in the normalized matrix by using the theory of NFRAAWA operator, such as

$$\mathfrak{I}^*{}_1 = \big((0.620535,0.419487,0.719439), (0.520492,0.819265,0.719439)\big)$$
$$\mathfrak{I}^*{}_2 = \big((0.720638,0.719439,0.519514), (0.921777,0.519514,0.819265)\big)$$
$$\mathfrak{I}^*{}_3 = \big((0.820882,0.619502,0.918601), (0.720638,0.719439,0.519514)\big)$$
$$\mathfrak{I}^*{}_4 = \big((0.620535,0.918601,0.619502), (0.620535,0.519514,0.719439)\big)$$
$$\mathfrak{I}^*{}_5 = \big((0.820882,0.519514,0.619502), (0.820882,0.719439,0.619502)\big)$$

Step 3: Evaluate the score values of the above-aggregated information, such as

$$\mathfrak{I}^*{}_1 = -0.5112, \mathfrak{I}^*{}_2 = -0.31177, \mathfrak{I}^*{}_3 = -0.41184, \mathfrak{I}^*{}_4 = -0.51199, \mathfrak{I}^*{}_5 = -0.27873$$

Step 4: Under consideration of the above information, we rank all the alternatives and find the best one, such as

$$\mathfrak{I}^*{}_5 > \mathfrak{I}^*{}_2 > \mathfrak{I}^*{}_3 > \mathfrak{I}^*{}_1 > \mathfrak{I}^*{}_4$$

The best optimal is $\mathfrak{I}^*{}_5$ represented the Structure Information. Furthermore, we check the stability and influence of the proposed techniques with the parameters. By using the data in Table 1, the influence of the parameter is stated in Table 2.

Table 2. Ranking values for different values of parameters.

$\equiv$	**Score value**	**Ranking result**
2	$-0.5112, -0.31177, -0.41184, -0.51199, -0.27873$	$\mathfrak{I}^*{}_5 > \mathfrak{I}^*{}_2 > \mathfrak{I}^*{}_3 > \mathfrak{I}^*{}_1 > \mathfrak{I}^*{}_4$
4	$-0.51051, -0.3101\ 8, -0.40995, -0.51014, -0.27738$	$\mathfrak{I}^*{}_5 > \mathfrak{I}^*{}_2 > \mathfrak{I}^*{}_3 > \mathfrak{I}^*{}_4 > \mathfrak{I}^*{}_1$
6	$-0.50921, -0.30863, -0.40818, -0.50842, -0.27604$	$\mathfrak{I}^*{}_5 > \mathfrak{I}^*{}_2 > \mathfrak{I}^*{}_3 > \mathfrak{I}^*{}_4 > \mathfrak{I}^*{}_1$

8	$-0.50777, -0.30712, -0.40655, -0.50682, -0,27437$	$\mathfrak{I}^*_5 > \mathfrak{I}^*_2 > \mathfrak{I}^*_3 > \mathfrak{I}^*_4 > \mathfrak{I}^*_1$
10	$-0.50639, -0.30567, -0.40504, -0.50535, -0.27345$	$\mathfrak{I}^*_5 > \mathfrak{I}^*_2 > \mathfrak{I}^*_3 > \mathfrak{I}^*_4 > \mathfrak{I}^*_1$

The best optimal is $\mathfrak{I}^*_5$ represented the Structure Information. Furthermore, we started the comparison of the proposed techniques with some prevailing operators to improve the worth of the derived operators.

6 | COMPARATIVE ANALYSIS

In this section, we compare the developed operators with some prevailing information based on the data in Table 1 to evaluate the supremacy and superiority of the derived techniques. Moreover, comparative analysis is a necessary part of every manuscript, because for shows the validity and reliability of the proposed techniques. For this, we use the following operators based on some existing ideas, such as: averaging operators for IFSs [22], and geometric operators for IFSs [23] based on Aczel-Alsina norms. Additionally, weighted operators for neutrosophic Z-number [25], aggregation operators for single-valued NSs [26], weighted aggregation operators for simplified NSs [27], power operators for IFSs [28], averaging operators for IFRSs [29], and prioritized operators for IFRSs [30] based on the Aczel-Alsina operational laws [30]. Considering the data in Table 1, the comparative analysis is listed in Table 3.

Table 3. Comparative analysis for proposed and existing techniques.

Methods	Score values	Ranking values
Senapati et al. [22]	$--bound\mathbb{E}xpd--$	$--bound\mathbb{E}xpd--$
Senapati et aal. [23]	$--bound\mathbb{E}xpd--$	$--bound\mathbb{E}xpd--$
Ye et al. [25]	$--bound\mathbb{E}xpd--$	$--bound\mathbb{E}xpd--$
Senapati [26]	$--bound\mathbb{E}xpd--$	$--bound\mathbb{E}xpd--$
Yong et al. [27]	$--bound\mathbb{E}xpd--$	$--bound\mathbb{E}xpd--$
Senapati et al. [28]	$--bound\mathbb{E}xpd--$	$--bound\mathbb{E}xpd--$
Ahmmad et al. [29]	$--bound\mathbb{E}xpd--$	$--bound\mathbb{E}xpd--$
Khan et al. [30]	$--bound\mathbb{E}xpd--$	$--bound\mathbb{E}xpd--$
Latif et al. [31]	$--bound\mathbb{E}xpd--$	$--bound\mathbb{E}xpd--$
NFRAAWA operator	$-0.5112, -0.31177, -0.41184, -0.51199, -0.27873$	$\mathfrak{I}^*_5 > \mathfrak{I}^*_2 > \mathfrak{I}^*_3 > \mathfrak{I}^*_1 > \mathfrak{I}^*_4$

The best optimal is $\mathfrak{I}^*_5$ represented the Structure Information. Moreover, we noticed that the existing techniques are not able to evaluate the information in Table 1, because these operators are computed based on the special cases of the proposed techniques. Hence, the evaluated operators are very dominant and reliable for depicting unreliable and vague information.

7 | CONCLUSION

The major contribution of this manuscript is listed below:

1. We derived the novel theory of Neutrosophic fuzzy rough sets and their operational laws.
2. We evaluated the Aczel-Alsina operational laws based on the NFR information.
3. We derived the NFRAAWA operator, NFRAAOWA operator, and NFRAAHA operator. Some suitable properties for the above operators are also invented in detail.
4. We found the major principle of the X-ray analysis of crystal structure; we illustrated the procedure of MADM techniques based on the invented operators for NSs.
5. We investigated the comparative analysis between proposed operators with some prevailing techniques to evaluate the supremacy and validity of the derived operators.

In the future, we aim to expose some new ideas based on the NFRSs and their relation extensions such as hesitant fuzzy sets and complex numbers. Furthermore, we will evaluate some practical applications in the field of artificial intelligence, neural networks, game theory, and neural networks to enhance the worth and depth of the proposed information.

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