ORIGINAL ARTICLE

OPEN ACCESS

Two-Phase Fluid Transport Phenomena in Phloem Vessels

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Article History:

Submitted: September 08, 2023
Revised: December 11, 2023
Accepted: December 27, 2023
Published online: December 30, 2023

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A B S T R A C T

In this study the two-phase (particle-fluid) Newtonian fluid is studied when flowing through sieve tube. The model of phloem is assumed with permeable walls. The flow is important for the study of transport of food in plants. Leaves prepare food using sunlight and minerals transported from roots to leaves using xylem cells which operates predominantly on the concept of Poiseuille flow. The pressure is generated due to processes of transpiration. While prepared food is transported from leaves to stem for storage and other parts of plant body to perform desired operations. Phloem operates with the concept of diffusion. Each phloem cell is a long tube with a permeable membrane at each end, which helps to keep the mineral fixed in food and transported to different parts of plant. Reflection coefficient is an important parameter which helps to show effects of permeability of diffusion membrane. Equations are modelled and solved using perturbation and HAM Solution are displayed graphical form.

Keywords: Two-phase, phloem, permeable walls, Poiseuille flow, diffusion, perturbation and HAM.

(MSC2020) Codes: 94D05, 28E10

1 | INTRODUCTION

The ascent of sap means the upward movement of water and minerals in plants that is from the roots to the leaves. There are two flow vessels in plants i) xylem vessels, driven by transpiration from the stomata in leaves, ii) Phloem from root to leave. The phloem, on the other hand, plays a crucial role in the transport of sugars and amino acids, which letter helps in producing food in the leaves through photosynthesis. These nutrients move bidirectionally within the phloem, supporting growth, energy storage, and various metabolic processes throughout the plant.

The literature review on phloem sap transport provides a comprehensive examination of the mechanisms and regulatory factors involved in the movement of phloem sap. It explores the evolutionary dimensions of plant development, the key metabolic aspects governing phloem sap transport, and the biochemical composition of the sap. The review also addresses challenges associated with phloem sap sampling and highlights unresolved questions in phloem transport biology.

Furthermore, the review encompasses widely accepted mechanisms of phloem transport, including osmotically generated pressure flow and the relay hypothesis. These mechanisms play a crucial role in controlling the speed and direction of solute movement along the phloem transport pathway. It will also discuss the importance of sugars as the principal components involved in phloem transport and the role of the phloem tissue as the principal sugar conductive tissue in plants [1]. Moreover, the review will shed light on the challenges associated with analyzing the biochemical composition of phloem sap. This is attributed to the sensitivity of sieve elements to physical injuries, as well as the difficulties in obtaining representative and uncontaminated samples of phloem sap for thorough analyses. It will also address the need for a detailed understanding of the metabolic composition of sap and the essential role of metabolite transport via phloem sap in plant growth and development [2]. Additionally, the literature review will investigate recent advancements and lingering inquiries in the field of phloem transport biology. This encompasses examining the relationships between structure and function that contribute to the bulk flow of sap, understanding the mechanisms of phloem transport in trees, and exploring the connections between transport processes and the demand at sink sites [3]-[4]. The review will offer a summary of the cellular pathways and molecular trafficking associated with phloem transport. This includes the conveyance of nutrients, defensive compounds, and signaling molecules across vascular plants [5]. Some other important papers are found in literature are found in [6], [7], [8] and [9] and therein.

The Homotopy Analysis Method (HAM) is a powerful mathematical technique for solving nonlinear problems, which have gained significant attention in recent years due to its applicability to a wide range of fields, including fluid mechanics, heat transfer, and the solution of initial value problems with discontinuities [10]-[11]. The homotopy analysis method was developed by Professor Shijun Liao in 1992 and has since been extensively used for solving nonlinear problems [12]. The method is based on the concept of homotopy, which involves the deformation of a given problem into a similar, but simpler, problem that can be solved more easily. This technique has been applied to various types of problems, including fluid mechanics, heat transfer, and the solution of initial value problems with discontinuities [10]. The homotopy analysis method has been used to study the flow of fluids through porous media, the motion of viscous fluids, and the behavior of free surface flows [13], [14], [15], [16], [17] and [18]. These studies have contributed to a better understanding of the underlying physical mechanisms and have led to the development of new theories and techniques for solving fluid mechanics problems. The homotopy analysis method has been applied to initial value problems of the second order with various types of discontinuities, including those involving finite discontinuities, distributional discontinuities, and discontinuities in higher-order derivatives [19]. These studies have demonstrated the versatility of the method and its ability to handle complex problems with discontinuities. The homotopy analysis method can be combined with other techniques, such as the perturbation method and homotopy in

topology, to obtain exact solutions or power series expansions for a wide range of problems [20], [21], [22], [23] and [24]. This coupling has allowed researchers to obtain more accurate and efficient solutions for various types of problems.

This study investigates the flow of a two-phase Newtonian fluid (particle-fluid) through a permeable-walled sieve tube model of the phloem, crucial for understanding food transport in plants. The model considers the movement of nutrients from leaves, where food is synthesized, to various plant parts. The phloem, operating on the diffusion concept, utilizes cells with permeable membranes, and the reflection coefficient is employed to assess membrane permeability effects. The study formulates and solves equations using perturbation and HAM methods, presenting results graphically.

2 | MATHEMATICAL FORMULATIONS

The equations for mass, momentum and concentration for fluid phase and particulate transfer cylindrical coordinates with azimuth coordinate is taken as zero are;

$$\frac{1}{r}\frac{\partial(rU_r)}{\partial r}+\frac{\partial U_z}{\partial z}=0, \tag{1}$$

$$\rho\left(U_r\frac{\partial U_r}{\partial r}+U_z\frac{\partial U_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+\overset{*}{\mu}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rU_r)}{\partial r}\right)+\frac{\partial^2 U_r}{\partial z^2}\right]+\frac{CS}{1-C}(U_{rp}-U_r) \tag{2}$$

$$\rho_f\left(U_r\frac{\partial U_z}{\partial r}+U_z\frac{\partial U_z}{\partial z}\right)=-\frac{\partial p}{\partial r}+\overset{*}{\mu}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rU_z)}{\partial r}\right)+\frac{\partial^2 U_z}{\partial z^2}\right]+\frac{CS}{1-C}(U_{zp}-U_z) \tag{3}$$

$$U_r\frac{\partial C}{\partial r}+U_z\frac{\partial C}{\partial z}=D\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right)+\frac{\partial^2 C}{\partial z^2}\right]+\frac{D_s C}{1-C}(C_p-C\ \ ), \tag{4}$$

$$\frac{1}{r}\frac{\partial(rU_{rp})}{\partial r}+\frac{\partial U_{zp}}{\partial z}=0 \tag{5}$$

$$\rho_p\left(U_{rp}\frac{\partial U_{rp}}{\partial r}+U_{zp}\frac{\partial U_{rp}}{\partial z}\right)=-C\frac{\partial p}{\partial r}+\frac{CS}{1-C}(U_z-U_{zp}) \tag{6}$$

$$\rho_p\left(U_{rp}\frac{\partial U_{zp}}{\partial r}+U_{zp}\frac{\partial U_{zp}}{\partial z}\right)=-C\frac{\partial p}{\partial r}+\frac{CS}{1-C}(U_z-U_{zp}) \tag{7}$$

$$U_{rp}\frac{\partial C_p}{\partial r}+U_{zp}\frac{\partial C_p}{\partial z}=D\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C_p}{\partial r}\right)\right]+\frac{D_s C}{1-C}(C\ \ -C_p) \tag{8}$$

Here subscript "p" shows the variable for particulate phase. Where the concentration drag is D_s .

In fluid phase and particulate phase boundary conditions are below.

Water and solute exchange in the sieve tube membrane with the help of the following equations. The radial flow of solution is obtained by using Starling's equation, across the membrane at (r=R),

$$U_r(R,z)=L_p p(R,z)-L_p p_{out}(z)-\sigma L_p R_g T[C(R,z)-C_{out}(z)], \tag{9}$$

$$U_{rp}(R,z)=L_p p(R,z)-L_{p_{out}}(z)-\sigma L_p R_g T\left[C_p(R,z)-C_{out}(z)\right] \tag{10}$$

$\prod = R_g TC$ is called Van't Hoff equation. Here, $L_p$is membrane hydraulic conductivity which is constant and $p_{out}(z)$is hydrostatic pressure and apoplast concentration is$C_{out}(z)$. It does not depend on r and assume linear variation with distance, hence

$$p_{out}(z) = p_{out}(0) + \frac{dp_{out}}{dz} z, \tag{11}$$

$$C_{out}(z) = C_{out}(0) + \frac{dC_{out}}{dz} z \tag{12}$$

However, hydrostatic pressure and apoplast concentration are not affected anywhere. Theσreflects coefficient of membrane of the sieve tube. We can define radial flux $J_r^s$and J_{rp}^s in the tubes.

$$J_r^s(R,z) = U_r(R,z)\bar{C}(z) - \sigma U_r(R,z)\bar{C}(z) + P_s[C(R,z) - C_{out}(z)]., \tag{13}$$

$$J_{rp}^s(R,z) = U_{rp}(R,z)\bar{C_p}(z) - \sigma U_{rp}(R,z)\bar{C_p}(z) + P_s\left[C_p(R,z) - C_{out}(z)\right]. \tag{14}$$

$\tilde{C}(z)$ and $\tilde{C_p}(z)$, which denote the solute's average concentration for fluid and particulate phase, is represented by $\tilde{C}(z) = \frac{C(R,z)+C_{out}(z)}{2}$ and $\tilde{C_p}(z) = \frac{C_p(R,z)+C_{out}(z)}{2}$ in the sieve tube membrane.

For no slip condition,

$$U_z(R,z) = 0, \tag{15}$$

$$U_{zp}(R,z) = 0 \tag{16}$$

Because Centre of the tube is symmetry point, at $r = 0$, which implies that U_r, U_{rp} and the radial derivatives of the U_z, U_{zp} and C, C_p are zero.

$$U_r(0,z) = 0, \frac{\partial U_z}{\partial r}(0,z) = 0, \frac{\partial C}{\partial r}(0,z) = 0, \tag{17}$$

$$U_{rp}(0,z) = 0, \frac{\partial U_{zp}}{\partial r}(0,z) = 0, \frac{\partial C_p}{\partial r}(0,z) = 0, \tag{18}$$

3 | SOLUTION METHODOLOGY

The equations and boundary conditions are non-dimensionalized using.

$$\left.\begin{aligned} &\hat{\bar{r}} = \frac{r}{R}, \hat{\bar{z}} = \frac{z}{L}, \hat{\bar{p}} = p_i \frac{R^2}{8LU\overset{*}{\mu}}, \hat{\bar{U}_r} = U_r \frac{L}{RU}, \hat{\bar{L}_p} = L_p \frac{16L^2\overset{*}{\mu}}{R^3}, \hat{\bar{P}_s} = P_s \frac{2L}{RU} \\ &, \hat{\bar{U}_z} = \frac{U_z}{U}, \hat{\bar{C}} = \frac{C}{c_i}, \hat{\bar{U}_{rp}} = U_{rp} \frac{L}{RU}, N = \frac{SR^2}{\overset{*}{\mu}}, \hat{\bar{U}_{zp}} = \frac{U_{zp}}{U} \text{ and } \hat{\bar{C}_p} = \frac{C_p}{c_i} \end{aligned}\right\} \tag{19}$$

The equations (3.1) to (3.8) becomes,

$$\frac{1}{\hat{\bar{r}}} \frac{\partial(\hat{\bar{r}}\hat{\bar{U}_r})}{\partial \hat{\bar{r}}} + \frac{\partial \hat{\bar{U}_z}}{\partial \hat{\bar{z}}} = 0 \tag{20}$$

$$Re\quad \varepsilon^3\left(\overset{\wedge}{U_r}\frac{\partial\overset{\wedge}{U_r}}{\partial\hat{r}}+\overset{\wedge}{U_z}\frac{\partial\overset{\wedge}{U_r}}{\partial\hat{z}}\right)=-8\frac{\partial\hat{p}}{\partial\hat{r}}+\varepsilon^2\left[\begin{matrix}\frac{\partial}{\partial\hat{r}}\left(\frac{1}{\hat{r}}\frac{\partial(\hat{r}\overset{\wedge}{U_r})}{\partial\hat{r}}\right)+\varepsilon^2\frac{\partial^2\overset{\wedge}{U_r}}{\partial\hat{z}^2}\\+\frac{C}{1-C}N\left(\overset{\wedge}{U_{rp}}-\overset{\wedge}{U_r}\right)\end{matrix}\right] \tag{21}$$

$$\begin{aligned}Re\quad \varepsilon\left(\overset{\wedge}{U_r}\frac{\partial\overset{\wedge}{U_z}}{\partial\hat{r}}+\overset{\wedge}{U_z}\frac{\partial\overset{\wedge}{U_z}}{\partial\hat{z}}\right)&\\=-8\frac{\partial\hat{p}}{\partial\hat{z}}+\frac{1}{\hat{r}}\frac{\partial}{\partial\hat{r}}\left(\hat{r}\frac{\partial\overset{\wedge}{U_z}}{\partial\hat{r}}\right)+\varepsilon^2\frac{\partial^2\overset{\wedge}{U_z}}{\partial^2\hat{z}}+\frac{C}{1-C}N\left(\overset{\wedge}{U_{zp}}-\overset{\wedge}{U_z}\right)&\end{aligned} \tag{22}$$

$$\varepsilon\left(\overset{\wedge}{U_r}\frac{\partial\hat{C}}{\partial\hat{r}}+\overset{\wedge}{U_z}\frac{\partial\hat{C}}{\partial\hat{z}}\right)=\frac{1}{Pe_r}\left[\frac{1}{\hat{r}}\frac{\partial}{\partial\hat{r}}\left(\hat{r}\frac{\partial\hat{C}}{\partial\hat{r}}\right)+\varepsilon^2\frac{\partial^2\hat{C}}{\partial^2\hat{z}}\right]+N_1\left(\overset{\wedge}{C_p}-\hat{C}\right) \tag{23}$$

$$\frac{1}{\hat{r}}\frac{\partial(\hat{r}\overset{\wedge}{U_{rp}})}{\partial\hat{r}}+\frac{\partial\overset{\wedge}{U_{zp}}}{\partial\hat{z}}=0 \tag{24}$$

$$Re\quad \varepsilon^3\left(\overset{\wedge}{U_{rp}}\frac{\partial\overset{\wedge}{U_{rp}}}{\partial\hat{r}}+\overset{\wedge}{U_{zp}}\frac{\partial\overset{\wedge}{U_{rp}}}{\partial\hat{z}}\right)=-8C\frac{\partial\hat{p}}{\partial\hat{r}}+\varepsilon^2CN\left(\overset{\wedge}{U_r}-\overset{\wedge}{U_{rp}}\right) \tag{25}$$

$$Re\quad \varepsilon\left(\overset{\wedge}{U_{rp}}\frac{\partial\overset{\wedge}{U_{zp}}}{\partial\hat{r}}+\overset{\wedge}{U_{zp}}\frac{\partial\overset{\wedge}{U_{zp}}}{\partial\hat{z}}\right)=-8C\frac{\partial\hat{p}}{\partial\hat{z}}+CN\left(\overset{\wedge}{U_{zp}}-\overset{\wedge}{U_z}\right) \tag{26}$$

$$\varepsilon\left(\overset{\wedge}{U_{rp}}\frac{\partial\overset{\wedge}{C_p}}{\partial\hat{r}}+\overset{\wedge}{U_{zp}}\frac{\partial\overset{\wedge}{C_p}}{\partial\hat{z}}\right)=\frac{1}{Pe_r}\frac{1}{\hat{r}}\frac{\partial}{\partial\hat{r}}\left(\hat{r}\frac{\partial\hat{C}}{\partial\hat{r}}\right)+N_1\left(\hat{C}-\overset{\wedge}{C_p}\right) \tag{27}$$

In dimensionless notation, the boundary conditions in fluid and particulate phase equations (9), (10) and (13) to (18) are

$$\overset{\wedge}{U_r}\left(1,\hat{z}\right)=\frac{\overset{\wedge}{L_p}}{2}\hat{p}\left(1,\hat{z}\right)-\frac{\overset{\wedge}{L_p}}{2}\overset{\wedge}{p_{out}}\left(\hat{z}\right)-\sigma\frac{\overset{\wedge}{L_p}}{2}\hat{H}\left[\hat{C}\left(1,\hat{z}\right)-\hat{C}_{out}\left(\hat{z}\right)\right]. \tag{28}$$

$$\frac{1}{Pe_r}\frac{\partial\hat{C}}{\partial\hat{r}}=\varepsilon\hat{U}_r\left(1,\hat{z}\right)\left[\hat{C}\left(1,\hat{z}\right)-(1-\sigma)\overset{\wedge}{\bar{C}}(\hat{z})\right]-\varepsilon\frac{\overset{\wedge}{P_s}}{2}\left[\hat{C}\left(1,\hat{z}\right)-\hat{C}_{out}\left(\hat{z}\right)\right], \tag{29}$$

$$\overset{\wedge}{U_z}\left(1,\hat{z}\right)=0 \tag{30}$$

$$\overset{\wedge}{U_r}\left(0,\hat{z}\right)=0,\frac{\partial\overset{\wedge}{U_r}}{\partial\hat{r}}\left(0,\hat{z}\right)=0,\ \frac{\partial\hat{C}}{\partial\hat{r}}\left(0,\hat{z}\right)=0. \tag{31}$$

$$\overline{\overline{U_{rp}}}\left(1,\overline{\overline{z}}\right)=\frac{\overline{\overline{L_p}}}{2}\left\{\overline{\overline{p}}\left(1,\overline{\overline{z}}\right)-\overline{\overline{p_{out}}}\left(\overline{\overline{z}}\right)-\sigma\overline{\overline{H}}\left[\overline{\overline{C_p}}\left(1,\overline{\overline{z}}\right)-\overline{\overline{C}}_{out}\left(\overline{\overline{z}}\right)\right]\right\} \tag{32}$$

$$\frac{1}{Pe_r}\frac{\partial\overline{\overline{C_p}}}{\partial\overline{\overline{r}}}=\varepsilon\overline{\overline{U}}_{rp}\left(1,\overline{\overline{z}}\right)\left[\overline{\overline{C_p}}\left(1,\overline{\overline{z}}\right)-(1-\sigma)\overline{\overline{C}}(\overline{\overline{z}})\right]-\varepsilon\frac{\overline{\overline{P_s}}}{2}\left[\overline{\overline{C_p}}\left(1,\overline{\overline{z}}\right)-\overline{C_{out}}\left(\overline{\overline{z}}\right)\right], \tag{33}$$

$$\overline{\overline{U_{zp}}}\left(1,\overline{\overline{z}}\right)=0 \tag{34}$$

$$\overline{\overline{U_{rp}}}\left(0,\overline{\overline{z}}\right)=0,\frac{\partial\overline{\overline{U_{rp}}}}{\partial\overline{\overline{r}}}\left(0,\overline{\overline{z}}\right)=0,\frac{\partial\overline{\overline{C_p}}}{\partial\overline{\overline{r}}}\left(0,\overline{\overline{z}}\right)=0. \tag{35}$$

3.2.1 | Perturbation Solution

The non-dimensional equations along with the defined boundary conditions are solved using perturbation method.

$$\left.\begin{aligned}U_r&=U_{r0}+\varepsilon^1U_{r1}+\ldots,\\U_z&=U_{z0}+\varepsilon^1U_{z1}+\ldots,\\C&=C_0+\varepsilon^1C_1+\ldots,\\U_{rp}&=U_{rp0}+\varepsilon^1U_{rp1}+\ldots,\\U_{zp}&=U_{zp0}+\varepsilon^1U_{zp1}+\ldots,\\C_p&=C_{p0}+\varepsilon^1C_{p1}+\ldots,\end{aligned}\right\} \tag{36}$$

Use series of equations (36) in equations (20) to (22) and (24) to (26) and comparing the equations for the order of ε, we get two sets of problem.

Zeroth Order problem

In the zeroth approximation, the terms of order ε^0 are considered. Linear partial differential equations for $\overline{\overline{U}}_{r0},\overline{\overline{U}}_{rp0},\overline{\overline{U}}_{z0},\overline{\overline{U}}_{zp0},\overline{\overline{C}}_0,\overline{\overline{C}}_{p0}$ and $\overline{\overline{p}}_0$ are

$$\frac{\partial\overline{\overline{p_0}}}{\partial\overline{\overline{r}}}=0 \tag{37}$$

$$-8\frac{\partial\overline{\overline{p_0}}}{\partial\overline{\overline{z}}}+\frac{1}{\overline{\overline{r}}}\frac{\partial}{\partial\overline{\overline{r}}}\left(\overline{\overline{r}}\frac{\partial\overline{\overline{U_{z0}}}}{\partial\overline{\overline{r}}}\right)+\frac{CN}{1-C}\left(\overline{\overline{U_{zp0}}}-\overline{\overline{U_{z0}}}\right)=0 \tag{38}$$

$$-8C\frac{\partial\overline{\overline{p_0}}}{\partial\overline{\overline{z}}}+CN\left(\overline{\overline{U}}_{z0}-\overline{\overline{U}}_{zp0}\right)=0 \tag{39}$$

$$\frac{1}{Pe_r}\left[\frac{1}{\overline{\overline{r}}}\frac{\partial}{\partial\overline{\overline{r}}}\left(\overline{\overline{r}}\frac{\partial\overline{\overline{C}}_{p0}}{\partial\overline{\overline{r}}}\right)\right]+N_1\left(\overline{\overline{C}}_0-\overline{\overline{C}}_{p0}\right)=0 \tag{40}$$

The corresponding solution of the above equations becomes

$$\overset{\wedge}{U_{z0}} = -\frac{\partial^2 \overset{\wedge}{p_0}}{\partial \overset{\wedge^2}{z}}\left(\overset{\wedge^2}{r} + \frac{2C}{1-C}\overset{\wedge^2}{r} - 1 - \frac{C}{1-C}\right) \tag{41}$$

$$\overset{\wedge}{U_{r0}} = -\frac{\partial^2 \overset{\wedge}{p_0}}{\partial \overset{\wedge^2}{z}}\left(\overset{\wedge^3}{r} - 2\hat{r} + \frac{2C}{1-C}\overset{\wedge^3}{r} - \frac{2C}{1-C}\hat{r}\right) \tag{42}$$

$$\hat{U}_{zp0} = 2\frac{\partial \overset{\wedge}{p_0}}{\partial \hat{z}}\left(-\frac{4}{N} + \frac{C}{1-C}\left(2\overset{\wedge^2}{r} - 1\right) + \left(\overset{\wedge^2}{r} - 1\right)\right) \tag{43}$$

$$\overset{\wedge}{U_{rp0}} = -\frac{\partial^2 \overset{\wedge}{p_0}}{\partial \overset{\wedge^2}{z}}\left(\overset{\wedge^3}{r} - 2\hat{r} + \frac{2C}{1-C}\overset{\wedge^3}{r} - \frac{2C}{1-C}\hat{r}\right) \tag{44}$$

Solving the concentration equations (23) and (27) for $\varepsilon = 0$ together with B.C, is $\overset{\wedge}{C_0}\left(\hat{z}\right), \overset{\wedge}{C_{p0}}\left(\hat{z}\right)$ to be independent of radial coordinate:

$$\frac{\partial \overset{\wedge}{C_0}}{\partial \hat{r}} = 0 \tag{45}$$

$$\frac{\partial \overset{\wedge}{C_{p0}}}{\partial \hat{r}} = 0 \tag{46}$$

First Order Problem

In the zeroth approximation, the terms of order ε^1 are considered. Linear partial differential equations for $\overset{\wedge}{U_{r0}}, \overset{\wedge}{U_{rp0}}, \overset{\wedge}{U_{z0}}, \overset{\wedge}{U_{zp0}}, \overset{\wedge}{C_0}, \overset{\wedge}{C_{p0}}$ and $\overset{\wedge}{p_0}$ are

$$\frac{\partial \overset{\wedge}{p_1}}{\partial \hat{r}} = 0 \tag{47}$$

$$Re \quad \varepsilon\left(\overset{\wedge}{U_{rp0}}\frac{\partial \overset{\wedge}{U_{zp0}}}{\partial \hat{r}} + \overset{\wedge}{U_{zp0}}\frac{\partial \overset{\wedge}{U_{zp0}}}{\partial \hat{z}}\right) = -8C\frac{\partial \overset{\wedge}{p_1}}{\partial \hat{z}} + CN\left(\overset{\wedge}{U_{z1}} - \overset{\wedge}{U_{zp1}}\right) \tag{48}$$

$$\left(\overset{\wedge}{U_{rp0}}\frac{\partial \overset{\wedge}{C_{p0}}}{\partial \hat{r}} + \overset{\wedge}{U_{zp0}}\frac{\partial \overset{\wedge}{C_{p0}}}{\partial \hat{z}}\right) = \frac{1}{Pe_r}\left[\frac{1}{\hat{r}}\frac{\partial}{\partial \hat{r}}\left(\hat{r}\frac{\partial \overset{\wedge}{C_{p1}}}{\partial \hat{r}}\right) + N_1\left(\overset{\wedge}{C_1} - \overset{\wedge}{C_{p1}}\right)\right] \tag{49}$$

$$Re\left(\overset{\wedge}{U_{r0}}\frac{\partial \overset{\wedge}{U_{z0}}}{\partial \hat{r}} + \overset{\wedge}{U_{z0}}\frac{\partial \overset{\wedge}{U_{z0}}}{\partial \hat{z}}\right) = -8\frac{\partial \overset{\wedge}{p_1}}{\partial \hat{z}} + \frac{1}{\hat{r}}\frac{\partial}{\partial \hat{r}}\left(\hat{r}\frac{\partial \overset{\wedge}{U_{z1}}}{\partial \hat{r}}\right) + \frac{CN}{1-C}\left(\overset{\wedge}{U_{zp1}} - \overset{\wedge}{U_{z1}}\right) \tag{50}$$

The corresponding solution of the above equations becomes

$$\overset{\wedge}{U_{r1}} = Re\begin{bmatrix} -2\dfrac{\partial}{\partial\hat{z}}\left(\dfrac{\partial^2\overset{\wedge}{p_0}}{\partial\overset{\wedge 2}{z}}\dfrac{\partial\overset{\wedge}{p_0}}{\partial\hat{z}}\right)\left(\begin{matrix}\dfrac{1}{120}\overset{\wedge 7}{r} - \dfrac{1}{18}\overset{\wedge 5}{r} - \dfrac{C}{30(1-C)}\overset{\wedge 7}{r} - \dfrac{C}{6(1-C)}\overset{\wedge 5}{r} \\ + \dfrac{C^2}{6(1-C)^2}\overset{\wedge 5}{r}\end{matrix}\right) \\ -4\dfrac{\partial}{\partial\hat{z}}\left(\dfrac{\partial^2\overset{\wedge}{p_0}}{\partial\overset{\wedge 2}{z}}\dfrac{\partial\overset{\wedge}{p_0}}{\partial\hat{z}}\right)\left(\begin{matrix}\dfrac{1}{240}\overset{\wedge 7}{r} - \dfrac{1}{30}\overset{\wedge 5}{r} \\ + \dfrac{C}{1-C}\left(\dfrac{1}{60}\overset{\wedge 7}{r} - \dfrac{1}{12}\overset{\wedge 5}{r} + \dfrac{1}{4}\overset{\wedge 3}{r}\right) \\ + \dfrac{C^2}{(1-C)^2}\left(\dfrac{1}{60}\overset{\wedge 7}{r} - \dfrac{1}{18}\overset{\wedge 5}{r} + \dfrac{1}{8}\overset{\wedge 3}{r}\right)\end{matrix}\right) \end{bmatrix}$$

$$-4\frac{\partial^2\overset{\wedge}{p_1}}{\partial\overset{\wedge 2}{z}}\left(1+\frac{C}{1-C}\right)\overset{\wedge 3}{r}$$

$$+\frac{Re}{(1-C)\begin{bmatrix} -2\dfrac{\partial}{\partial\hat{z}}\left(\dfrac{\partial^2\overset{\wedge}{p_0}}{\partial\overset{\wedge 2}{z}}\dfrac{\partial\overset{\wedge}{p_0}}{\partial\hat{z}}\right)\left(\begin{matrix}\left(\dfrac{C}{1-C}\right)\left(\dfrac{-1}{60}\overset{\wedge 7}{r} - \dfrac{1}{35}\overset{\wedge 6}{r} + \dfrac{2}{9}\overset{\wedge 5}{r} - 4\overset{\wedge 3}{r}\right) \\ - \dfrac{2C^2}{3(1-C)^2}\overset{\wedge 2}{r} + \dfrac{4}{N^2}\overset{\wedge 3}{r} \\ + \dfrac{C}{N(1-C)}\left(\dfrac{2}{9}\overset{\wedge 5}{r} - \overset{\wedge 3}{r}\right) \\ + \dfrac{16}{N}\left(\dfrac{1}{72}\overset{\wedge 5}{r} - \dfrac{1}{8}\overset{\wedge 3}{r}\right) \\ + \left(-\dfrac{1}{120}\overset{\wedge 7}{r} + \dfrac{1}{70}\overset{\wedge 6}{r} - \dfrac{1}{18}\overset{\wedge 5}{r} - \dfrac{1}{4}\overset{\wedge 3}{r}\right)\end{matrix}\right) \end{bmatrix}}$$

$$-Re\begin{bmatrix} 2\dfrac{\partial}{\partial\hat{z}}\left(\dfrac{\partial^2\overset{\wedge}{p_0}}{\partial\overset{\wedge 2}{z}}\dfrac{\partial\overset{\wedge}{p_0}}{\partial\hat{z}}\right)\left(-\dfrac{2}{15}\hat{r} - \dfrac{11C}{30(1-C)}\hat{r} + \dfrac{C^2}{3(1-C)^2}\hat{r}\right) \\ -4\dfrac{\partial}{\partial\hat{z}}\left(\dfrac{\partial^2\overset{\wedge}{p_0}}{\partial\overset{\wedge 2}{z}}\dfrac{\partial\overset{\wedge}{p_0}}{\partial\hat{z}}\right)\left(-\dfrac{1}{15}\hat{r} + \dfrac{19C}{30(1-C)}\hat{r} + \dfrac{3C^2}{20(1-C)^2}\hat{r}\right) \end{bmatrix}$$

$$+2\frac{\partial^2\overset{\wedge}{p_1}}{\partial\overset{\wedge 2}{z}}\left(1+\frac{C}{1-C}\right)\hat{r}$$

$$+\frac{Re}{1-C\begin{bmatrix} -2\dfrac{\partial}{\partial\hat{z}}\left(\dfrac{\partial^2\overset{\wedge}{p_0}}{\partial\overset{\wedge 2}{z}}\dfrac{\partial\overset{\wedge}{p_0}}{\partial\hat{z}}\right)\left(\begin{matrix}-\dfrac{15C}{2(1-C)}\hat{r} - \dfrac{C^2}{(1-C)^2}\hat{r} + \dfrac{8}{N}\hat{r} \\ -\dfrac{4C}{3N(1-C)}\hat{r} - \dfrac{C}{3N}\hat{r} - \dfrac{13}{20}\hat{r}\end{matrix}\right) \end{bmatrix}}$$

(51)

$$\overset{\wedge}{U_{z1}} = Re\begin{bmatrix} -2\dfrac{\partial^2\overset{\wedge}{p_0}}{\partial\overset{\wedge 2}{z}}\dfrac{\partial\overset{\wedge}{p_0}}{\partial\hat{z}}\left(\begin{matrix}\dfrac{1}{15}\overset{\wedge 6}{r} - \dfrac{1}{3}\overset{\wedge 4}{r} + \dfrac{4C}{15(1-C)}\overset{\wedge 6}{r} - \dfrac{C}{(1-C)}\overset{\wedge 4}{r} + \dfrac{3C^2}{(1-C)^2}\overset{\wedge 4}{r} \\ + \dfrac{11C}{15(1-C)} - \dfrac{2C^2}{3(1-C)^2} + \dfrac{4}{15}\end{matrix}\right) \\ +4\dfrac{\partial^2\overset{\wedge}{p_0}}{\partial\overset{\wedge 2}{z}}\dfrac{\partial\overset{\wedge}{p_0}}{\partial\hat{z}}\left(\begin{matrix}\dfrac{1}{30}\overset{\wedge 6}{r} - \dfrac{1}{6}\overset{\wedge 4}{r} - \dfrac{C}{1-C}\left(\dfrac{1}{15}\overset{\wedge 6}{r} - \dfrac{1}{2}\overset{\wedge 4}{r} + \overset{\wedge 2}{r}\right) \\ + \dfrac{C^2}{(1-C)^2}\left(\dfrac{2}{15}\overset{\wedge 6}{r} - \dfrac{1}{3}\overset{\wedge 4}{r} + \dfrac{1}{2}\overset{\wedge 2}{r} + \dfrac{19}{30(1-C)} + \dfrac{3C^2}{10(1-C)^2}\right)\end{matrix}\right) \end{bmatrix}$$

(52)

$$+4\frac{\partial\overline{\overline{p}}_1}{\partial\overline{\overline{z}}}\left[\left(1+\frac{C}{1-C}\right)\overline{\overline{r}}^2-\left(1+\frac{C}{1-C}\right)\right]$$

$$-\frac{1}{1-C}Re\left[2\frac{\partial^2\overline{\overline{p}}_0}{\partial\overline{\overline{z}}^2}\frac{\partial\overline{\overline{p}}_0}{\partial\overline{\overline{z}}}\left(\begin{array}{l}-\frac{2C}{15(1-C)}\overline{\overline{r}}^6-\frac{C}{5(1-C)}\overline{\overline{r}}^5\\+\frac{4C}{3(1-C)}\overline{\overline{r}}^4-\frac{16C}{1-C}\overline{\overline{r}}^2\\-\frac{2C^2}{(1-C)^2}\overline{\overline{r}}+\frac{16}{N^2}\overline{\overline{r}}^2\\+\frac{C}{N(1-C)}\left(\frac{4}{3}\overline{\overline{r}}^4-4\overline{\overline{r}}^2\right)\\+\frac{4}{3N}\left(\overline{\overline{r}}^4-6\overline{\overline{r}}^2\right)\\\left(-\frac{1}{15}\overline{\overline{r}}^6+\frac{1}{10}\overline{\overline{r}}^5-\frac{1}{3}\overline{\overline{r}}^4-\overline{\overline{r}}^2\right)\\-\frac{15C}{1-C}-\frac{2C^2}{(1-C)^2}+\frac{16}{N^2}\\-\frac{8C}{3N(1-C)}+\frac{20}{3N}-\frac{13}{10}\end{array}\right)\right]$$

$$\overline{U_{zp1}}=-\frac{8}{N}\frac{\partial\overline{\overline{p}}_1}{\partial\overline{\overline{z}}}+\mathrm{Re}\left[\begin{array}{l}-2\frac{\partial^2\overline{\overline{p}}_0}{\partial\overline{\overline{z}}^2}\frac{\partial\overline{\overline{p}}_0}{\partial\overline{\overline{z}}}\left(\begin{array}{l}\frac{1}{15}\overline{\overline{r}}^6-\frac{1}{3}\overline{\overline{r}}^4+\frac{4C}{15(1-C)}\overline{\overline{r}}^6-\frac{C}{1-C}\overline{\overline{r}}^4\\+\frac{2C^2}{3(1-C)^2}\overline{\overline{r}}^4\end{array}\right)\\+4\frac{\partial^2\overline{\overline{p}}_0}{\partial\overline{\overline{z}}^2}\frac{\partial\overline{\overline{p}}_0}{\partial\overline{\overline{z}}}\left(\begin{array}{l}\frac{1}{30}\overline{\overline{r}}^6-\frac{1}{6}\overline{\overline{r}}^4+\frac{C}{1-C}\left(\begin{array}{l}\frac{2}{15}\overline{\overline{r}}^6\\-\frac{1}{2}\overline{\overline{r}}^4+\overline{\overline{r}}^2\end{array}\right)\\+\frac{C^2}{(1-C)^2}\left(\frac{2}{15}\overline{\overline{r}}^6-\frac{1}{3}\overline{\overline{r}}^4+\frac{1}{2}\overline{\overline{r}}^2\right)\end{array}\right)\end{array}\right] \tag{53}$$

$$+4\frac{\partial \overline{\overline{p_1}}}{\partial \overline{\overline{z}}}\left(1+\frac{C}{1-C}\right)\overline{\overline{r}}^2-\frac{\mathrm{Re}}{1-C}\left[2\frac{\partial^2 \overline{\overline{p_0}}}{\partial \overline{\overline{z}}^2}\frac{\partial \overline{\overline{p_0}}}{\partial \overline{\overline{z}}}\left(\begin{array}{l}\frac{C}{1-C}\left(\begin{array}{l}-\frac{2}{15}\overline{\overline{r}}^6-\frac{1}{15}\overline{\overline{r}}^5\\+\frac{4}{3}\overline{\overline{r}}^4-16\overline{\overline{r}}^2\end{array}\right)\\-\frac{2C^2}{(1-C)^2}\overline{\overline{r}}+\frac{16}{N^2}\overline{\overline{r}}^2\\+\frac{C}{N(1-C)}\left(\frac{4}{3}\overline{\overline{r}}^4-4\overline{\overline{r}}^2\right)\\+\frac{16}{N}\left(\frac{1}{12}\overline{\overline{r}}^4-\frac{1}{2}\overline{\overline{r}}^2\right)\\+\left(-\frac{1}{15}\overline{\overline{r}}^6+\frac{1}{10}\overline{\overline{r}}^5-\frac{1}{3}\overline{\overline{r}}^4-\overline{\overline{r}}^2\right)\end{array}\right)\right]$$

$$-\mathrm{Re}\left[\begin{array}{l}-2\frac{\partial^2 \overline{\overline{p_0}}}{\partial \overline{\overline{z}}^2}\frac{\partial \overline{\overline{p_0}}}{\partial \overline{\overline{z}}}\left(-\frac{4}{15}-\frac{11C}{15(1-C)}+\frac{2C^2}{3(1-C)^2}\right)\\+4\frac{\partial^2 \overline{\overline{p_0}}}{\partial \overline{\overline{z}}^2}\frac{\partial \overline{\overline{p_0}}}{\partial \overline{\overline{z}}}\left(-\frac{2}{15}+\frac{19C}{30(1-C)}+\frac{3C^2}{10(1-C)^2}\right)\end{array}\right]$$

$$-4\frac{\partial \overline{\overline{p_1}}}{\partial \overline{\overline{z}}}\left(1+\frac{C}{1-C}\right)-\frac{\mathrm{Re}}{1-C}\left[2\frac{\partial^2 \overline{\overline{p_0}}}{\partial \overline{\overline{z}}^2}\frac{\partial \overline{\overline{p_0}}}{\partial \overline{\overline{z}}}\left(\begin{array}{l}-\frac{15C}{1-C}-\frac{2C^2}{(1-C)^2}\\+\frac{16}{N^2}-\frac{8C}{3N(1-C)}\\-\frac{20}{3N}-\frac{13}{10}\end{array}\right)\right]$$

$$-\frac{\mathrm{Re}}{CN}\left[\begin{array}{l} -2\dfrac{\partial^2 \overset{\wedge}{p_0}}{\partial \hat{z}^2}\dfrac{\partial \overset{\wedge}{p_0}}{\partial \hat{z}}\left(\begin{array}{l} \dfrac{8C}{1-C}\hat{r}^4-\dfrac{12C}{1-C}\hat{r}^2 \\ +2\hat{r}^4-4\hat{r}^2+\dfrac{8C^2}{(1-C)^2}\hat{r}^4 \\ -\dfrac{8C^2}{(1-C)^2}\hat{r}^2 \end{array}\right) \\ +4\dfrac{\partial^2 \overset{\wedge}{p_0}}{\partial \hat{z}^2}\dfrac{\partial \overset{\wedge}{p_0}}{\partial \hat{z}}\left(\begin{array}{l} \dfrac{16}{N^2}+\dfrac{8}{N}-\dfrac{8}{N}\hat{r}^2-\dfrac{16C}{N(1-C)}\hat{r}^2 \\ +\dfrac{8C}{N(1-C)}+\dfrac{4C^2}{(1-C)^2}\hat{r}^4 \\ -\dfrac{4C^2}{(1-C)^2}\hat{r}^2+\dfrac{4C}{1-C}\hat{r}^4 \\ -\dfrac{6C}{1-C}\hat{r}^2+\dfrac{C^2}{(1-C)^2}+\dfrac{2C}{1-C} \\ +\hat{r}^4-2\hat{r}^2+1 \end{array}\right) \end{array}\right]$$

3.2.2 | Pressure and Concentration Profile

To calculate the pressure profiles, we use equations (28) and (32) replacing the perturbation solution of zeroth and first order of velocities equations become

$$\frac{\partial^2 \overset{\wedge}{\overline{p_0}}}{\partial \overset{\wedge}{\overline{z}}^2}=\overset{\wedge}{\overline{L_p}}\left\{\overset{\wedge}{\overline{p_0}}\left(\overset{\wedge}{\overline{z}}\right)-\overset{\wedge}{\overline{p_{out}}}\left(\overset{\wedge}{\overline{z}}\right)-\sigma\overset{\wedge}{\overline{H}}\left[\overset{\wedge}{\overline{C_0}}\left(1,\overset{\wedge}{\overline{z}}\right)-\overset{\wedge}{\overline{C_{out}}}\left(\overset{\wedge}{\overline{z}}\right)\right]\right\} \tag{54}$$

$$2\frac{\partial^2\hat{\overline{p_1}}}{\partial\hat{\overline{z}}^2}\left(1+\frac{C}{1-C}\right)=Re\begin{bmatrix}-2\frac{\partial}{\partial\hat{\overline{z}}}\left(\frac{\partial^2\hat{\overline{p_0}}}{\partial\hat{\overline{z}}^2}\frac{\partial\hat{\overline{p_0}}}{\partial\hat{\overline{z}}}\right)\left(-\frac{17}{360}-\frac{C}{5(1-C)}+\frac{C^2}{6(1-C)^2}\right)\\-4\frac{\partial}{\partial\hat{\overline{z}}}\left(\frac{\partial^2\hat{\overline{p_0}}}{\partial\hat{\overline{z}}^2}\frac{\partial\hat{\overline{p_0}}}{\partial\hat{\overline{z}}}\right)\left(-\frac{17}{720}+\frac{11C}{60(1-C)}+\frac{31C^2}{360(1-C)^2}\right)\end{bmatrix}$$

$$+\frac{Re}{(1-C)\left[-2\frac{\partial}{\partial\hat{\overline{z}}}\left(\frac{\partial^2\hat{\overline{p_0}}}{\partial\hat{\overline{z}}^2}\frac{\partial\hat{\overline{p_0}}}{\partial\hat{\overline{z}}}\right)\left(\begin{matrix}\frac{-4817C}{1260(1-C)}-\frac{2C^2}{3(1-C)^2}+\frac{4}{N^2}\\+\frac{7C}{9N(1-C)}+\frac{16}{9N}-\frac{151}{504}\end{matrix}\right)\right]}$$

$$-Re\begin{bmatrix}2\frac{\partial}{\partial\hat{\overline{z}}}\left(\frac{\partial^2\hat{\overline{p_0}}}{\partial\hat{\overline{z}}^2}\frac{\partial\hat{\overline{p_0}}}{\partial\hat{\overline{z}}}\right)\left(-\frac{2}{15}-\frac{11C}{30(1-C)}+\frac{C^2}{3(1-C)^2}\right)\\-4\frac{\partial}{\partial\hat{\overline{z}}}\left(\frac{\partial^2\hat{\overline{p_0}}}{\partial\hat{\overline{z}}^2}\frac{\partial\hat{\overline{p_0}}}{\partial\hat{\overline{z}}}\right)\left(-\frac{1}{15}+\frac{19C}{30(1-C)}+\frac{3C^2}{20(1-C)^2}\right)\end{bmatrix} \tag{55}$$

$$+\frac{Re}{1-C\left[-2\frac{\partial}{\partial\hat{\overline{z}}}\left(\frac{\partial^2\hat{\overline{p_0}}}{\partial\hat{\overline{z}}^2}\frac{\partial\hat{\overline{p_0}}}{\partial\hat{\overline{z}}}\right)\left(\begin{matrix}-\frac{15C}{2(1-C)}-\frac{C^2}{(1-C)^2}+\frac{8}{N}\\-\frac{4C}{3N(1-C)}-\frac{C}{3N}-\frac{13}{20}\end{matrix}\right)\right]}$$

$$-\frac{\hat{\overline{L_p}}}{2}\left\{\hat{\overline{p_0}}\left(\hat{\overline{z}}\right)-\hat{\overline{p_{out}}}\left(\hat{\overline{z}}\right)-\sigma\hat{\overline{H}}\left[\hat{\overline{C_1}}\left(1,\hat{\overline{z}}\right)-\hat{\overline{C_{out}}}\left(\hat{\overline{z}}\right)\right]\right\}$$

$$\frac{\partial^2\hat{\overline{p_0}}}{\partial\hat{\overline{z}}^2}=\hat{\overline{L_p}}\left\{\hat{\overline{p_0}}\left(\hat{\overline{z}}\right)-\hat{\overline{p_{out}}}\left(\hat{\overline{z}}\right)-\sigma\hat{\overline{H}}\left[\hat{\overline{C_{p0}}}\left(1,\hat{\overline{z}}\right)-\hat{\overline{C_{out}}}\left(\hat{\overline{z}}\right)\right]\right\} \tag{56}$$

$$\frac{\partial^2 \overset{\wedge}{p_1}}{\partial \hat{z}^2}\left(\frac{4}{N}+\frac{3C}{1-C}+3\right)=\mathrm{Re}\left[\begin{array}{l}-2\frac{\partial}{\partial \hat{z}}\left(\frac{\partial^2 \overset{\wedge}{p_0}}{\partial \hat{z}^2}\frac{\partial \overset{\wedge}{p_0}}{\partial \hat{z}}\right)\left(-\frac{17}{360}+\frac{2C}{45(1-C)}+\frac{C^2}{9\left(1-C^2\right)}\right)\\+\frac{\partial}{\partial \hat{z}}\left(\frac{\partial^2 \overset{\wedge}{p_0}}{\partial \hat{z}^2}\frac{\partial \overset{\wedge}{p_0}}{\partial \hat{z}}\right)\left(\frac{-17}{720}+\frac{8C}{45(1-C)}+\frac{19C^2}{360(1-C)^2}\right)\end{array}\right]$$

$$-\frac{\mathrm{Re}}{1-C}\left[2\frac{\partial}{\partial \hat{z}}\left(\frac{\partial^2 \overset{\wedge}{p_0}}{\partial \hat{z}^2}\frac{\partial \overset{\wedge}{p_0}}{\partial \hat{z}}\right)\left(\begin{array}{l}\frac{-4817C}{1260(1-C)}-\frac{2C^2}{(1-C)^2}+\frac{4}{N^2}\\+\frac{7C}{9N(1-C)}-\frac{16}{9N}-\frac{151}{504}\end{array}\right)\right]$$

$$-\mathrm{Re}\left[\begin{array}{l}-2\frac{\partial}{\partial \hat{z}}\left(\frac{\partial^2 \overset{\wedge}{p_0}}{\partial \hat{z}^2}\frac{\partial \overset{\wedge}{p_0}}{\partial \hat{z}}\right)\left(-\frac{2}{15}-\frac{11C}{30(1-C)}+\frac{C^2}{3(1-C)^2}\right)\\+4\frac{\partial}{\partial \hat{z}}\left(\frac{\partial^2 \overset{\wedge}{p_0}}{\partial \hat{z}^2}\frac{\partial \overset{\wedge}{p_0}}{\partial \hat{z}}\right)\left(-\frac{1}{15}+\frac{19C}{60(1-C)}+\frac{3C^2}{20(1-C)^2}\right)\end{array}\right] \qquad (57)$$

$$-\frac{\mathrm{Re}}{1-C}\left[2\frac{\partial}{\partial \hat{z}}\left(\frac{\partial^2 \overset{\wedge}{p_0}}{\partial \hat{z}^2}\frac{\partial \overset{\wedge}{p_0}}{\partial \hat{z}}\right)\left(\begin{array}{l}-\frac{15C}{2(1-C)}-\frac{C^2}{(1-C)^2}-\frac{2}{N}\\-\frac{4C}{3N(1-C)}-\frac{13}{20}\end{array}\right)\right]$$

$$-\frac{\mathrm{Re}}{CN}\left[\begin{array}{l}-2\frac{\partial}{\partial \hat{z}}\left(\frac{\partial^2 \overset{\wedge}{p_0}}{\partial \hat{z}^2}\frac{\partial \overset{\wedge}{p_0}}{\partial \hat{z}}\right)\left(\frac{-5C}{3(1-C)}-\frac{2}{3}-\frac{2C^2}{3(1-C)^2}\right)\\+4\frac{\partial}{\partial \hat{z}}\left(\frac{\partial^2 \overset{\wedge}{p_0}}{\partial \hat{z}^2}\frac{\partial \overset{\wedge}{p_0}}{\partial \hat{z}}\right)\left(\begin{array}{l}\frac{8}{N^2}+\frac{2}{N}\hat{r}-\frac{4C^2}{N(1-C)^2}+\frac{4C}{N(1-C)}\\+\frac{C^2}{6(1-C)^2}+\frac{C}{6(1-C)}+\frac{1}{6}\end{array}\right)\end{array}\right]$$

$$+\frac{\overset{\wedge}{L_p}}{2}\left\{\overset{\wedge}{p_0}\left(\hat{z}\right)-\overset{\wedge}{p_{out}}(z)-\sigma\hat{H}\left[\overset{\wedge}{C_{p0}}\left(1,\hat{z}\right)-\overset{\wedge}{C_{out}}\left(\hat{z}\right)\right]\right\}$$

As zeroth order concentration profile revel that $\hat{C}_0$and $\hat{C}_{p0}$ are independent of radial coordinates equations (23) and (27) become

$$\frac{\partial \overset{\wedge}{C_1}}{\partial \hat{r}}=Pe_r\frac{\partial \overset{\wedge}{p_0}}{\partial \hat{z}}\frac{\partial \overset{\wedge}{C_0}}{\partial \hat{z}}\hat{r} \qquad (58)$$

$$\frac{\partial \overset{\wedge}{\overline{C}}_{p1}}{\partial \overset{\wedge}{\overline{r}}} = Pe_r \frac{\partial \overset{\wedge}{\overline{p_0}}}{\partial \overset{\wedge}{\overline{z}}} \frac{\partial \overset{\wedge}{\overline{C}}_{p0}}{\partial \overset{\wedge}{\overline{z}}} \left(-\frac{4}{N} \overset{\wedge}{\overline{r}} + \frac{C}{1-C} \overset{\wedge}{\overline{r}} \right) \tag{59}$$

From the equations (17) and (23), the first order expressions are given by,

$$\frac{1}{Pe_r} \frac{\partial \overset{\wedge}{\overline{C}}_1}{\partial \overset{\wedge}{\overline{r}}} \left(1, \overset{\wedge}{\overline{z}}\right) = \overset{\wedge}{\overline{U}}_{r0} \left(1, \overset{\wedge}{\overline{z}}\right) \left[\overset{\wedge}{\overline{C}}_0 \left(1, \overset{\wedge}{\overline{z}}\right) - (1-\sigma) \overset{\wedge}{\overline{\overline{C}}}_0 \left(\overset{\wedge}{\overline{z}}\right) \right] - \frac{\overset{\wedge}{\overline{P}}_s}{2} \left[\overset{\wedge}{\overline{C}}_0 \left(1, \overset{\wedge}{\overline{z}}\right) - \overset{\wedge}{\overline{C}}_{out} \left(\overset{\wedge}{\overline{z}}\right) \right] \tag{60}$$

$$\begin{aligned} \frac{1}{Pe_r} \frac{\partial \overset{\wedge}{\overline{C}}_{p1}}{\partial \overset{\wedge}{\overline{r}}} \left(1, \overset{\wedge}{\overline{z}}\right) &= \overset{\wedge}{\overline{U}}_{rp0} \left(1, \overset{\wedge}{\overline{z}}\right) \left[\overset{\wedge}{\overline{C}}_{p0} \left(1, \overset{\wedge}{\overline{z}}\right) - (1-\sigma) \overset{\wedge}{\overline{\overline{C}}}_{p0} \left(\overset{\wedge}{\overline{z}}\right) \right] \\ &- \frac{\overset{\wedge}{\overline{P}}_s}{2} \left[\overset{\wedge}{\overline{C}}_{p0} \left(1, \overset{\wedge}{\overline{z}}\right) - \overset{\wedge}{\overline{C}}_{out} \left(\overset{\wedge}{\overline{z}}\right) \right] \end{aligned} \tag{61}$$

Combining Equations (58) and (60) and by using equations (42) and (54), the zeroth order equation for concentration of fluid phase is given by

$$\begin{aligned} \frac{\partial \overset{\wedge}{\overline{p_0}}}{\partial \overset{\wedge}{\overline{z}}} \frac{\partial \overset{\wedge}{\overline{C}}_0}{\partial \overset{\wedge}{\overline{r}}} &= \left[\frac{\overset{\wedge}{\overline{L_p}}}{2} \left\{ \overset{\wedge}{\overline{p_0}} \left(\overset{\wedge}{\overline{z}}\right) - \overset{\wedge}{\overline{p_{out}}} \left(\overset{\wedge}{\overline{z}}\right) - \sigma \overset{\wedge}{\overline{H}} \left[\overset{\wedge}{\overline{C}}_0 \left(1, \overset{\wedge}{\overline{z}}\right) - \overset{\wedge}{\overline{C_{out}}} \left(\overset{\wedge}{\overline{z}}\right) \right] \right\} \right] * \\ &\left[\overset{\wedge}{\overline{C}}_0 - (1-\sigma) \overset{\wedge}{\overline{\overline{C}}}_0 \right] - \frac{\overset{\wedge}{\overline{P}}_s}{2} \left(\overset{\wedge}{\overline{C}}_0 - \overset{\wedge}{\overline{C}}_{out} \right) \end{aligned} \tag{62}$$

By combining equations (59) and (61) and by using equations (44) and (56), the zeroth order equation for concentration of particulate phase is given by

$$\begin{aligned} \frac{\partial \overset{\wedge}{\overline{p_0}}}{\partial \overset{\wedge}{\overline{z}}} \frac{\partial \overset{\wedge}{\overline{C}}_{p0}}{\partial \overset{\wedge}{\overline{r}}} \left(\frac{C}{1-C} - \frac{4}{N} \right) &= \left[\frac{\overset{\wedge}{\overline{L_p}}}{2} \left\{ \overset{\wedge}{\overline{p_0}} \left(\overset{\wedge}{\overline{z}}\right) - \overset{\wedge}{\overline{p_{out}}}(z) - \sigma \overset{\wedge}{\overline{H}} \left[\overset{\wedge}{\overline{C_{p0}}} \left(1, \overset{\wedge}{\overline{z}}\right) - \overset{\wedge}{\overline{C_{out}}} \left(\overset{\wedge}{\overline{z}}\right) \right] \right\} \right] * \\ &\left[\overset{\wedge}{\overline{C}}_{p0} - (1-\sigma) \overset{\wedge}{\overline{\overline{C}}}_{p0} \right] - \frac{\overset{\wedge}{\overline{P}}_s}{2} \left(\overset{\wedge}{\overline{C}}_{p0} - \overset{\wedge}{\overline{C}}_{out} \right) \end{aligned} \tag{63}$$

Using zeroth and first order solution in equations (58) and (59), we get

$$\overset{\wedge}{\overline{C}}_1 = \frac{Pe_r}{2} \frac{\partial \overset{\wedge}{\overline{p_0}}}{\partial \overset{\wedge}{\overline{z}}} \frac{\partial \overset{\wedge}{\overline{C}}_0}{\partial \overset{\wedge}{\overline{r}}} \overset{\wedge}{\overline{r}}^2 + \frac{Pe_r}{4} \frac{\partial \overset{\wedge}{\overline{C}}_0 \left(\overset{\wedge}{\overline{z}} = 0\right)}{\partial \overset{\wedge}{\overline{z}}} \tag{64}$$

$$\overset{\wedge}{\overline{C}}_{p1} = Pe_r \frac{\partial \overset{\wedge}{\overline{p_0}}}{\partial \overset{\wedge}{\overline{z}}} \frac{\partial \overset{\wedge}{\overline{C}}_{p0}}{\partial \overset{\wedge}{\overline{z}}} \left(-\frac{2}{N} \overset{\wedge}{\overline{r}}^2 + \frac{C}{2(1-C)} \overset{\wedge}{\overline{r}}^2 \right) + Pe_r \frac{\partial \overset{\wedge}{\overline{C}}_{p0} \left(\overset{\wedge}{\overline{z}} = 0\right)}{\partial \overset{\wedge}{\overline{z}}} \left(-\frac{2}{N} + \frac{C}{2(1-C)} \right) \tag{65}$$

Now the unknown variables, $\overset{\wedge}{C_0}, \overset{\wedge}{C_{p0}}, \overset{\wedge}{p_0}$ and $\overset{\wedge}{p_1}$ are only to be obtained. Due to the solution of the coupled differential equations (54)- (57), (62) and (63).

3.2.3 | HAM Solution

Equations (54)- (57), (62) and (63) are solved using HAM.

For series solution: initial guesses $p_{00}(z), p_{10}(z), C_{00}(z)$,Linear operators $\pounds_1(p_0), \pounds_2(p_1), \pounds_3(C_0)$ and auxiliary parameters $h_{p_0}, h_{p_1}, h_{C_0}$ of zeroth order pressure, first order pressure ,zeroth order concentration are defined respectively with inserting parameter $\eta \in [0,1]$.

$$p_{00}(z) = pi - z \quad , \quad p_{10}(z) = 0.3375(-Lp\,Re\,p\,i + LpRe\sigma H)z \, , \, C_{00}(z) = 1 - z. \tag{66}$$

$$\pounds_1 = p_0^{''}(z), \pounds_2 = p_1^{''}(z), \pounds_3 = C_0^{'}(z). \tag{67}$$

The zeroth order deformation problem is represented as:

$$\left.\begin{aligned}(1-\eta)\pounds_1[p_0(z,\eta) - p_{00}(z)] &= \eta h_{p_0} N_{p_0}[p_0(z,\eta), p_1(z,\eta), C_0(z,\eta)],\\ (1-\eta)\pounds_2[p_1(z,\eta) - p_{10}(z)] &= \eta h_{p_1} N_{p_1}[p_0(z,\eta), p_1(z,\eta), C_0(z,\eta)],\\ (1-\eta)\pounds_3[C_0(z,\eta) - C_{00}(z)] &= \eta h_{C_0} N_{C_0}[p_0(z,\eta), p_1(z,\eta), C_0(z,\eta)].\end{aligned}\right\} \tag{68}$$

$$\left.\begin{array}{lll}\text{For} & \eta = 0 & \eta = 1\\ p_0(z,\eta): & p_{00}(z) & p_0(z)\\ p_1(z,\eta): & p_{10}(z) & p_1(z)\\ C_0(z,\eta): & C_{00}(z) & C_0(z)\end{array}\right\} \tag{69}$$

N_{p_0}, N_{p_1} and N_{C_0} are non-linear operators:

$$
\begin{aligned}
&N_{p_0}\left[p_0(z,\eta),p_1(z,\eta),\mathrm{C}_0(z,\eta)\right]=\frac{\partial^2 p_0(z,\eta)}{\partial z^2}-L_p\begin{Bmatrix}p_0(z,\eta)-p_{out}\\-\sigma H\left[C_0(z,\eta)-C_{out}\right]\end{Bmatrix},\\
&N_{p_1}\left[p_0(z,\eta),p_1(z,\eta),\mathrm{C}_0(z,\eta)\right]=2\frac{\partial^2 p_1(z,\eta)}{\partial z^2}\left(1+\frac{c}{1-c}\right)\\
&-\mathrm{Re}\begin{bmatrix}-2\dfrac{\partial}{\partial z}\left(\dfrac{\partial^2 p_0(z,\eta)}{\partial z^2}\dfrac{\partial p_0(z,\eta)}{\partial z}\right)\left(-\dfrac{17}{360}-\dfrac{c}{5(1-c)}+\dfrac{c^2}{6(1-c)^2}\right)\\-4\dfrac{\partial}{\partial z}\left(\dfrac{\partial^2 p_0(z,\eta)}{\partial z^2}\dfrac{\partial p_0(z,\eta)}{\partial z}\right)\left(-\dfrac{17}{720}+\dfrac{11c}{60(1-c)}+\dfrac{31c^2}{360(1-c)^2}\right)\end{bmatrix}\\
&-\frac{\mathrm{Re}}{(1-c)}\left[-2\frac{\partial}{\partial z}\left(\frac{\partial^2 p_0(z,\eta)}{\partial z^2}\frac{\partial p_0(z,\eta)}{\partial z}\right)\left(\begin{matrix}\dfrac{-4817c}{1260(1-c)}-\dfrac{2c^2}{3(1-c)^2}+\dfrac{4}{N^2}\\+\dfrac{7c}{9N(1-c)}+\dfrac{16}{9N}-\dfrac{151}{504}\end{matrix}\right)\right]\\
&+\mathrm{Re}\begin{bmatrix}2\dfrac{\partial}{\partial z}\left(\dfrac{\partial^2 p_0(z,\eta)}{\partial z^2}\dfrac{\partial p_0(z,\eta)}{\partial z}\right)\left(-\dfrac{2}{15}-\dfrac{11c}{30(1-c)}+\dfrac{c^2}{3(1-c)^2}\right)\\-4\dfrac{\partial}{\partial z}\left(\dfrac{\partial^2 p_0(z,\eta)}{\partial z^2}\dfrac{\partial p_0(z,\eta)}{\partial z}\right)\left(-\dfrac{1}{15}+\dfrac{19c}{30(1-c)}+\dfrac{3c^2}{20(1-c)^2}\right)\end{bmatrix}\\
&-\frac{\mathrm{Re}}{1-c}\left[-2\frac{\partial}{\partial z}\left(\frac{\partial^2 p_0(z,\eta)}{\partial z^2}\frac{\partial p_0(z,\eta)}{\partial z}\right)\left(\begin{matrix}-\dfrac{15c}{2(1-c)}-\dfrac{c^2}{(1-c)^2}+\dfrac{8}{N}\\-\dfrac{4c}{3N(1-c)}-\dfrac{c}{3N}-\dfrac{13}{20}\end{matrix}\right)\right]\\
&+\frac{L_p}{2}\left\{p_0(z,\eta)-p_{out}-\sigma H\left[C_1(z,\eta)-C_{out}\right]\right\},\\
&N_{C_0}\left[p_0(z,\eta),p_1(z,\eta),\mathrm{C}_0(z,\eta)\right]=\frac{\partial p_0(z,\eta)}{\partial z}\frac{\partial C0(z,\eta)}{\partial r}\\
&-\left[\frac{L_p}{2}\left\{p_0(z,\eta)-p_{out}-\sigma H\left[C_0(z,\eta)-C_{out}\right]\right\}\right]\left[C0(z,\eta)-(1-\sigma)\bar{C}0(z,\eta)\right]\\
&+\frac{Ps}{2}\left(C0(z,\eta)-Cout\right).
\end{aligned}
\tag{70}
$$

When inserting parameter η varies from 0 to 1, then $p_0(z,\eta)$, $p_1(z,\eta)$ and $C_0(z,\eta)$ change form initial guesses $p_{00}(z)$,$p_{10}(z)$ and $C_{00}(z)$ to wanted $p_0(z)$,$p_1(z)$ and $C_1(z)$ solution:

Expansion of $p_0(z,\eta)$, $p_1(z,\eta)$ and $C_0(z,\eta)$ by using Taylor's series as

$$\left.\begin{aligned} p_0(\mathrm{z},\eta) &= p_{00}(z)+\sum_{k=1}^{\infty} p_{0k}(z)\eta^k, \\ p_1(\mathrm{z},\eta) &= p_{10}(z)+\sum_{k=1}^{\infty} p_{1k}(z)\eta^k, \\ C_0(\mathrm{z},\eta) &= C_{00}(z)+\sum_{k=1}^{\infty} C_{0k}(z)\eta^k. \end{aligned}\right\} \tag{71}$$

In which

$$p_{0k}(y)=\frac{1}{\mathrm{k!}}\frac{\partial^k p_0(\mathrm{z},\eta)}{\partial\eta^k}\bigg|_{\eta=0}, \mathrm{p}_{1k}(z)=\frac{1}{\mathrm{k!}}\frac{\partial^k p_1(\mathrm{z},\eta)}{\partial\eta^k}\bigg|_{\eta=0}, \mathrm{C}_{0k}(z)=\frac{1}{\mathrm{k!}}\frac{\partial^k C_0(\mathrm{z},\eta)}{\partial\eta^k}\bigg|_{\eta=0}. \tag{72}$$

Now differentiate k −times Eq. (68) to zero[th]-order deformation w.r.t η and result divide by $k!$ then use $\eta=0$ and obtained kth −order deformation expression for $p_{0l}(z)$, $p_{1l}(z)$ and $C_{0l}(z)$ as follow:

$$\left.\begin{aligned} £_1[p_{0k}(z)-\hat{\chi}_k p_{0k-1}(z)] &= \hbar_{p_0} R_k^{p_0}(z), \\ £_2[p_{1k}(z)-\hat{\chi}_k p_{1k-1}(z)] &= \hbar_{p_1} R_k^{p_1}(z), \\ £_3[C_{0k}(z)-\hat{\chi}_k C_{0k-1}(z)] &= \hbar_{C_0} R_k^{C_0}(z). \end{aligned}\right\} \tag{73}$$

$$\left.\begin{aligned} &p_{0k}(\mathrm{z},\eta)=p_i, \mathrm{p}_{1k}(z,\eta)=0, \mathrm{C}_{0k}(z,\eta)=1, p'_{0k}(\mathrm{z},\eta)=-1, \\ &\mathrm{p}'_{1k}(z,\eta)=-\frac{3re}{8}\left(L_p\left\{p_0(z,\eta)-p_{out}-\sigma H\left[C_0(z,\eta)-C_{out}\right]\right\}\right), at\ \mathrm{z=0}. \end{aligned}\right\} \tag{74}$$

$$\hat{\chi}_k=\begin{cases}0, & \mathrm{k}\le 1\\ 1, & \mathrm{k}>1\end{cases}. \tag{75}$$

$$
\left.
\begin{aligned}
&R_k^{p_0}(z) = p_{0k}^{''} - L_p\left\{p_{0k} - p_{out} - \sigma H\left[C_{0k} - C_{out}\right]\right\},\\
&R_k^{p_1}(z) = 2p_{1k}^{''}\left(1+\frac{c}{1-c}\right)\\
&-\mathrm{Re}\left[\begin{array}{l}
-2\left(\sum_{m=0}^{k} p_{0m}^{'''} p_{0(k-m)}^{'} - \sum_{m=0}^{k} p_{0m}^{''} p_{0(k-m)}^{''}\right)\left(-\frac{17}{360} - \frac{c}{5(1-c)} + \frac{c^2}{6(1-c)^2}\right)\\
-4\left(\sum_{m=0}^{k} p_{0m}^{'''} p_{0(k-m)}^{'} - \sum_{m=0}^{k} p_{0m}^{''} p_{0(k-m)}^{''}\right)\left(-\frac{17}{720} + \frac{11c}{60(1-c)} + \frac{31c^2}{360(1-c)^2}\right)
\end{array}\right]\\
&-\frac{\mathrm{Re}}{(1-c)}\left[-2\left(\sum_{m=0}^{k} p_{0m}^{'''} p_{0(k-m)}^{'} - \sum_{m=0}^{k} p_{0m}^{''} p_{0(k-m)}^{''}\right)\left(\begin{array}{l}
\frac{-4817c}{1260(1-c)} - \frac{2c^2}{3(1-c)^2} + \frac{4}{N^2}\\
+\frac{7c}{9N(1-c)} + \frac{16}{9N} - \frac{151}{504}
\end{array}\right)\right]\\
&+\mathrm{Re}\left[\begin{array}{l}
2\left(\sum_{m=0}^{k} p_{0m}^{'''} p_{0(k-m)}^{'} - \sum_{m=0}^{k} p_{0m}^{''} p_{0(k-m)}^{''}\right)\left(-\frac{2}{15} - \frac{11c}{30(1-c)} + \frac{c^2}{3(1-c)^2}\right)\\
-4\left(\sum_{m=0}^{k} p_{0m}^{'''} p_{0(k-m)}^{'} - \sum_{m=0}^{k} p_{0m}^{''} p_{0(k-m)}^{''}\right)\left(-\frac{1}{15} + \frac{19c}{30(1-c)} + \frac{3c^2}{20(1-c)^2}\right)
\end{array}\right]\\
&-\frac{\mathrm{Re}}{1-c}\left[-2\left(\sum_{m=0}^{k} p_{0m}^{'''} p_{0(k-m)}^{'} - \sum_{m=0}^{k} p_{0m}^{''} p_{0(k-m)}^{''}\right)\left(\begin{array}{l}
-\frac{15c}{2(1-c)} - \frac{c^2}{(1-c)^2} + \frac{8}{N}\\
-\frac{4c}{3N(1-c)} - \frac{c}{3N} - \frac{13}{20}
\end{array}\right)\right]\\
&+\frac{L_p}{2}\left\{p_{0k} - p_{out} - \sigma H\left[C_{1k}(z,\eta) - C_{out}\right]\right\},\\
&R_k^{C_0}(z) = \sum_{m=0}^{k} p_{0m}^{'} C_{0(k-m)}^{'}\\
&-\left[\frac{L_p}{2}\left\{p_{0k} - p_{out} - \sigma H\left[C_{0k} - C_{out}\right]\right\}\right]\left[C0k - (1-\sigma)\bar{C}0k\right]\\
&+\frac{Ps}{2}\left(C0k - Cout\right).
\end{aligned}
\right\} \qquad (76)
$$

The approximation for solution of *k*-order can be stated as

$$\left.\begin{aligned} p_0(z) &= p_{00}(z) + \sum_{m=0}^{k} p_{0m}(z), \\ p_1(z,) &= p_{10}(z) + \sum_{m=0}^{k} p_{1m}(z), \\ C_0(z) &= C_{00}(z) + \sum_{m=0}^{k} C_{0m}(z). \end{aligned}\right\} \tag{77}$$

The analytic solution of zeroth order pressure, first order pressure, zeroth order concentration up to second order approximations are obtained.

$$\begin{aligned} p_0(z) = \frac{3873}{500} - \frac{9999z}{10000} + hz^2\left[-\frac{1397}{1000} - \frac{9741\sigma}{10000} + h\left[-\frac{1397}{1000} - \frac{4871\sigma}{5000}\right]\right] + hz^3\left[\frac{73}{1000}\right. \\ \left. + \frac{3633\sigma}{10000} + h\left[\frac{73}{1000} + \frac{3633\sigma}{10000}\right]\right] + h^2z^4\left[\frac{221}{5000} - \frac{33\sigma}{400} - \frac{121\sigma^2}{1000} - \frac{197\sigma^3}{1000}\right] \\ + h^2z^5\left[-\frac{13}{10000} - \frac{11\sigma}{2000} + \frac{2859\sigma^2}{100000} + \frac{199\sigma^3}{2500}\right] + h^2z^6\sigma\left[\frac{3}{10000} - \frac{9\sigma}{10000}\right. \\ \left. - \frac{33\sigma^2}{2500}\right] \end{aligned} \tag{78}$$

$$p_1(z) = 0\frac{213hz^2}{10000} + \frac{27h^3z^2}{10000(-1+c)^3N^2} - \frac{9hz^3}{1000} + \frac{3621hz^3\sigma}{10000} - \frac{0.00003003c^2h^5z^4\sigma^2}{N^3c^6}$$
$$+ \frac{4281h^2z^2}{100000c^3} - \frac{107ch^2z^2}{1250c^3} + \frac{427c^2h^2z^2}{10000c^3} + \frac{c^3h^2z^2}{c^3} + \frac{h^3z^2}{100000c^3} + \frac{21ch^3z^2}{10000c^3}$$
$$- \frac{c^2h^3z^2}{1250c^3} - \frac{c^3h^3z^2}{1250c^3} + \frac{3h^2z^2}{20000N^2c^3} + \frac{7ch^3z^2}{1250N^2c^3} - \frac{7c^2h^3z^2}{2500N^2c^3} - \frac{h^2z^2}{2500\,N\,c^3}$$
$$+ \frac{ch^2z^2}{1250\,N\,c^3} + \frac{17h^2z^3}{250c^3} - \frac{113ch^2z^3}{500c^3} - \frac{11ch^3z^3}{50000c^3} + \frac{c^2h^3z^3}{10000\,N\,c^3} - \frac{17h^2z^4}{5000c^3}$$
$$+ \frac{13ch^2z^4}{1250c^3} - \frac{101c^2h^2z^4}{10000c^3} + \frac{17c^3h^2z^4}{5000c^3} + \frac{2.0579\times10^{-7}h^3z^4}{c^3} + \frac{ch^3z^4}{10000c^3}$$
$$- \frac{c^2h^3z^4}{10000c^3} - \frac{7c^4h^5z^4}{20000000(-1+c)^3N^2c^3} + \frac{9c^3h^2z^6}{10000c^3} + \frac{9553ch^2z^2\sigma}{2500c^3}$$
$$- \frac{487c^2h^2z^2\sigma}{250c^3} + \frac{3c^2h^2z^2\sigma}{5000N^2c^3} - \frac{19h^3z^2\sigma}{5000N^2c^3} - \frac{21c^2h^2z^2\sigma}{10000\,N\,c^3} + \frac{c^3h^2z^2\sigma}{5000\,N\,c^3}$$
$$+ \frac{17h^3z^2\sigma}{10000\,N\,c^3} + \frac{7883h^2z^3\sigma}{10000c^3} + \frac{2279c^2h^2z^3\sigma}{2500c^3} - \frac{617c^3h^2z^3\sigma}{10000c^3} - \frac{h^3z^3\sigma}{10000c^3}$$
$$- \frac{3ch^3z^3\sigma}{2500c^3} + \frac{c^2h^3z^3\sigma}{2000c^3} + \frac{c^3h^3z^3\sigma}{2000c^3} + \frac{h^3z^3\sigma}{1000N^2c^3} - \frac{33ch^3z^3\sigma}{10000N^2c^3} + \frac{c^2h^3z^3\sigma}{625N^2c^3}$$
$$+ \frac{391ch^2z^4\sigma}{1000c^3} - \frac{3919c^2h^2z^4\sigma}{10000c^3} + \frac{653c^3h^2z^4\sigma}{5000c^3} + \frac{h^3z^4\sigma}{10000c^3} + \frac{ch^3z^4\sigma}{10000c^3}$$
$$- \frac{c^2h^3z^4\sigma}{5000c^3} + \frac{h^3z^4\sigma}{5000\,N\,c^3} - \frac{ch^3z^4\sigma}{10000\,N\,c^3} + \frac{3c^2h^3z^4\sigma}{100000\,N\,c^3} + \frac{67h^2z^5\sigma}{50000c^3} \qquad (79)$$
$$- \frac{403ch^2z^5\sigma}{100000c^3} + \frac{c^2h^2z^5\sigma}{250c^3} - \frac{13c^3h^2z^5\sigma}{10000c^3} + \frac{h^2z^6\sigma}{2500c^3} - \frac{11ch^2z^6\sigma}{10000c^3} - \frac{c^2h^3z^2\sigma^2}{2500c^3}$$
$$+ \frac{27ch^3z^2\sigma^2}{10000N^2c^3} - \frac{9ch^3z^2\sigma^2}{10000\,N\,c^3} - \frac{h^3z^3\sigma^2}{2000\,N\,c^3} + \frac{7ch^3z^3\sigma^2}{10000\,N\,c^3} - \frac{c^2h^3z^3\sigma^2}{N\,c^3}$$
$$- \frac{1211h^2z^4\sigma^2}{10000c^3} + \frac{3631ch^2z^4\sigma^2}{10000c^3} - \frac{9079c^2h^2z^4\sigma^2}{25000c^3} - \frac{9h^2z^6\sigma^2}{10000c^3} + \frac{7ch^2z^6\sigma^2}{2500c^3}$$
$$- \frac{7c^2h^2z^6\sigma^2}{2500c^3} - \frac{197h^2z^4\sigma^3}{1000c^3} + \frac{ch^2z^4\sigma^3}{c^3} + \frac{239c^2h^2z^5\sigma^3}{1000c^3} - \frac{199c^3h^2z^5\sigma^3}{2500c^3}$$
$$- \frac{33h^2z^6\sigma^3}{2500c^3} + \frac{199ch^2z^6\sigma^3}{5000c^3} - \frac{199c^2h^2z^6\sigma^3}{5000c^3} + \frac{33c^3h^2z^6\sigma^3}{2500c^3} - \frac{h^3z^4\sigma0.c^3}{c^3}$$
$$+ \frac{0.00403ch^2z^5\sigma}{(-1.+c)^3} - \frac{0.004c^2h^2z^5\sigma}{(-1.+c)^3} + \frac{0.0013c^3h^2z^5\sigma}{(-1.+c)^3} - \frac{0.0004h^2z^6\sigma}{(-1.+c)^3}$$
$$+ \frac{0.0011ch^2z^6\sigma}{(-1.+c)^3} + \frac{0.0004c^2h^3z^2\sigma^2}{(-1.+c)^3} - \frac{0.0027ch^3z^2\sigma^2}{(-1.+c)^3N^2} + \frac{0.0007ch^3z^3\sigma^2}{(-1.+c)^3N}$$
$$+ \frac{c^2h^3z^3\sigma^2}{(-1.+c)^3N} + \frac{0.0028c^2h^2z^6\sigma^2}{(-1.+c)^3} - \frac{0.0009c^3h^2z^6}{(-1.+c)^3} + \frac{0.197h^2z^4\sigma^3}{(-1.+c)^3}$$
$$- \frac{ch^2z^4\sigma^3}{(-1.+c)^3} - \frac{0.239c^2h^2z^5\sigma^3}{(-1.+c)^3} + \frac{0.0796c^3h^2z^5\sigma^3}{(-1.+c)^3} + \frac{0.0132h^2z^6\sigma^3}{(-1.+c)^3}$$
$$- \frac{0.0398ch^2z^6\sigma^3}{(-1.+c)^3} + \frac{0.0398c^2h^2z^6\sigma^3}{(-1.+c)^3} - \frac{0.0132c^3h^2z^6\sigma^3}{(-1.+c)^3},$$

$$C_0(z) = 1 - z + \frac{6283\mathrm{hz}^2}{10000} - \frac{\mathrm{hz}^3}{80} - \frac{11\mathrm{hz}^4}{2000} + \frac{1677\mathrm{hz}^2\sigma}{2500} - \frac{331\mathrm{hz}^3\sigma}{1250} + \frac{129\mathrm{hz}^4\sigma}{10000} + \frac{10921\mathrm{hz}^2\sigma^2}{10000} - \frac{7359\mathrm{hz}^3\sigma^2}{10000} + \frac{919\mathrm{hz}^4\sigma^2}{5000}. \quad (80)$$

For series solution: initial guesses $p_{00}(z)$, $p_{10}(z)$, $C_{p00}(z)$,Linear operators $£_1(p_0), £_2(p_1), £_3(C_{p0})$ and auxiliary parameters $h_{p_0}, h_{p_1}, h_{C_{p0}}$of zeroth order pressure, first order pressure ,zeroth order concentration for particulate phase are defined respectively with inserting parameter $\eta \in [0,1]$.

$$p_{00}(z) = pi - z \;\; , \;\; p_{10}(z) = 0.3375(-Lp\,Re\,p\,i + LpRe\sigma H)z \; , \; C_{p00}(z) = 1 - z \quad (81)$$

$$£_1 = p_0{}''(z), £_2 = p_1{}''(z), £_3 = C_{p0}{}'(z) \quad (82)$$

The zeroth order deformation problem is represented as:

$$\left.\begin{aligned} (1-\eta)£_1[p_0(z,\eta) - p_{00}(z)] &= \eta h_{p_0} N_{p_0}[p_0(z,\eta), p_1(z,\eta), C_{p0}(z,\eta)], \\ (1-\eta)£_2[p_1(z,\eta) - p_{10}(z)] &= \eta h_{p_1} N_{p_1}[p_0(z,\eta), p_1(z,\eta), C_{p0}(z,\eta)], \\ (1-\eta)£_3[C_{p0}(z,\eta) - C_{p00}(z)] &= \eta h_{C_{p0}} N_{C_{p0}}[p_0(z,\eta), p_1(z,\eta), C_{p0}(z,\eta)]. \end{aligned}\right\} \quad (83)$$

$$\left.\begin{array}{lll} \text{For} & \eta = 0 & \eta = 1 \\ p_0(z,\eta): & p_{00}(z) & p_0(z) \\ p_1(z,\eta): & p_{10}(z) & p_1(z) \\ C_{p0}(z,\eta): & C_{p00}(z) & C_{p0}(z) \end{array}\right\} \quad (84)$$

N_{p_0}, N_{p_1} and $N_{C_{p0}}$ are non-linear operators:

$$
\begin{aligned}
&N_{p_0}\left[p_0(z,\eta),p_1(z,\eta),\mathrm{C}_{p0}(z,\eta)\right]=\frac{\partial^2 p_0(z,\eta)}{\partial z^2}-L_p\left\{\begin{matrix}p_0(z,\eta)-p_{out}\\-\sigma H\left[C_{p0}(z,\eta)-C_{out}\right]\end{matrix}\right\},\\
&N_{p_1}\left[p_0(z,\eta),p_1(z,\eta),\mathrm{C}_{p0}(z,\eta)\right]=2\frac{\partial^2 p_1(z,\eta)}{\partial z^2}\left(\frac{4}{N}+\frac{3c}{1-c}+3\right)\\
&-\mathrm{Re}\left[\begin{matrix}-2\frac{\partial}{\partial \hat{z}}\left(\frac{\partial^2 p_0(z,\eta)}{\partial z^2}\frac{\partial p_0(z,\eta)}{\partial z}\right)\left(-\frac{17}{360}-\frac{c}{45(1-c)}+\frac{c^2}{6(1-c)^2}\right)\\-4\frac{\partial}{\partial z}\left(\frac{\partial^2 p_0(z,\eta)}{\partial z^2}\frac{\partial p_0(z,\eta)}{\partial z}\right)\left(-\frac{17}{720}+\frac{11c}{60(1-c)}+\frac{31c^2}{360(1-c)^2}\right)\end{matrix}\right]\\
&+\frac{\mathrm{Re}}{(1-c)}\left[-2\frac{\partial}{\partial z}\left(\frac{\partial^2 p_0(z,\eta)}{\partial z^2}\frac{\partial p_0(z,\eta)}{\partial z}\right)\left(\begin{matrix}\frac{-4817c}{1260(1-c)}-\frac{2c^2}{3(1-c)^2}+\frac{4}{N^2}\\+\frac{7c}{9N(1-c)}+\frac{16}{9N}-\frac{151}{504}\end{matrix}\right)\right]\\
&+\mathrm{Re}\left[\begin{matrix}2\frac{\partial}{\partial z}\left(\frac{\partial^2 p_0(z,\eta)}{\partial z^2}\frac{\partial p_0(z,\eta)}{\partial z}\right)\left(-\frac{2}{15}-\frac{11c}{30(1-c)}+\frac{c^2}{3(1-c)^2}\right)\\-4\frac{\partial}{\partial z}\left(\frac{\partial^2 p_0(z,\eta)}{\partial z^2}\frac{\partial p_0(z,\eta)}{\partial z}\right)\left(-\frac{1}{15}+\frac{19c}{30(1-c)}+\frac{3c^2}{20(1-c)^2}\right)\end{matrix}\right]\\
&-\frac{\mathrm{Re}}{1-c}\left[-2\frac{\partial}{\partial z}\left(\frac{\partial^2 p_0(z,\xi)}{\partial z^2}\frac{\partial p_0(z,\xi)}{\partial z}\right)\left(\begin{matrix}-\frac{15c}{2(1-c)}-\frac{c^2}{(1-c)^2}+\frac{8}{N}\\-\frac{4c}{3N(1-c)}-\frac{c}{3N}-\frac{13}{20}\end{matrix}\right)\right]\\
&+\frac{L_p}{2}\left\{p_0(z,\eta)-p_{out}-\sigma H\left[C_{p1}(z,\eta)-C_{out}\right]\right\},\\
&N_{C_{p0}}\left[p_0(z,\eta),p_1(z,\eta),\mathrm{C}_{p0}(z,\eta)\right]=\frac{\partial p_0(z,\eta)}{\partial z}\frac{\partial C_{p0}(z,\eta)}{\partial r}\\
&-\left[\frac{L_p}{2}\left\{p_0(z,\eta)-p_{out}-\sigma H\left[C_{p0}(z,\eta)-C_{out}\right]\right\}\right]\left[C_{p0}(z,\eta)-(1-\sigma)\bar{C}_{p0}(z,\eta)\right]\\
&+\frac{P_S}{2}\left(C_{p0}(z,\eta)-Cout\right).
\end{aligned}
\qquad (85)
$$

When inserting parameter η varies from 0 to 1, then $p_0(z,\eta)$, $p_1(z,\eta)$ and $C_{p0}(z,\eta)$ change form initial guesses $p_{00}(z)$,$p_{10}(z)$ and $C_{p00}(z)$ to wanted $p_0(z)$,$p_1(z)$ and $C_{p1}(z)$ solution:

Expansion of $p_0(z,\eta)$, $p_1(z,\eta)$ and $C_{p0}(z,\eta)$ by using Taylor's series as

$$\left.\begin{aligned} p_0(z,\eta) &= p_{00}(z) + \sum_{k=1}^{\infty} p_{0k}(z)\eta^k, \\ p_1(z,\eta) &= p_{10}(z) + \sum_{k=1}^{\infty} p_{1k}(z)\eta^k, \\ C_{p0}(z,\eta) &= C_{p00}(z) + \sum_{k=1}^{\infty} C_{p0_k}(z)\eta^k. \end{aligned}\right\} \tag{86}$$

In which

$$p_{0k}(z) = \frac{1}{k!}\frac{\partial^k p_0(z,\eta)}{\partial \eta^k}\bigg|_{\eta=0}, \; p_{1k}(z) = \frac{1}{k!}\frac{\partial^k p_1(z,\eta)}{\partial \eta^k}\bigg|_{\eta=0}, \tag{87}$$

$$C_{p0k}(z) = \frac{1}{k!}\frac{\partial^k C_{p0}(z,\eta)}{\partial \eta^k}\bigg|_{\eta=0}.$$

Now differentiate k −times Eq. (83) to zeroth-order deformation w.r.t η and result divide by $k!$ then use $\eta = 0$ and obtained kth −order deformation expression for $p_{0l}(z)$, $p_{1l}(z)$ and $C_{p0l}(z)$ as follow:

$$\left.\begin{aligned} £_1[p_{0k}(z) - \hat{\chi}_k p_{0k-1}(z)] &= \hbar_{p_0} R_k^{p_0}(z), \\ £_2[p_{1k}(z) - \hat{\chi}_k p_{1k-1}(z)] &= \hbar_{p_1} R_k^{p_1}(z), \\ £_3\left[C_{p0k}(z) - \hat{\chi}_k C_{p0k-1}(z)\right] &= \hbar_{C_{p0}} R_k^{C_{p0}}(z). \end{aligned}\right\} \tag{88}$$

$$\left.\begin{aligned} &p_{0k}(z,\eta) = p_i, p_{1k}(z,\eta) = 0, C_{p0l}(z,\eta) = 1, p'_{0k}(z,\eta) = -1, \\ &p'_{1k}(z,\eta) = -\frac{3re}{8}\left(L_p\begin{Bmatrix} p_0(z,\eta) - p_{out} \\ -\sigma H\left[C_{p0}(z,\eta) - C_{out}\right] \end{Bmatrix}\right), at \; z=0. \end{aligned}\right\} \tag{89}$$

$$\hat{\chi}_k = \begin{cases} 0, & k \le 1 \\ 1, & k > 1 \end{cases}. \tag{90}$$

$$
\left.
\begin{aligned}
&R_k^{p_0}(z) = p''_{0k} - L_p\left\{p_{0k} - p_{out} - \sigma H\left[C_{p0k} - C_{out}\right]\right\},\\
&R_k^{p_1}(z) = 2p''_{1k}\left(\frac{4}{N} + \frac{3c}{1-c} + 3\right)\\
&-\mathrm{Re}\left[\begin{aligned}
&-2\left(\sum_{m=0}^{k} p'''_{0m}p'_{0(k-m)} - \sum_{m=0}^{k} p''_{0m}p''_{0(k-m)}\right)\left(-\frac{17}{360} - \frac{c}{5(1-c)} + \frac{c^2}{6(1-c)^2}\right)\\
&-4\left(\sum_{m=0}^{k} p'''_{0m}p'_{0(k-m)} - \sum_{m=0}^{k} p''_{0m}p''_{0(k-m)}\right)\left(-\frac{17}{720} + \frac{11c}{60(1-c)} + \frac{31c^2}{360(1-c)^2}\right)
\end{aligned}\right]\\
&+\frac{\mathrm{Re}}{(1-c)}\left[-2\left(\sum_{m=0}^{k} p'''_{0m}p'_{0(k-m)} - \sum_{m=0}^{k} p''_{0m}p''_{0(k-m)}\right)\left(\begin{aligned}
&\frac{-4817c}{1260(1-c)} - \frac{2c^2}{3(1-c)^2} + \frac{4}{N^2}\\
&+\frac{7c}{9N(1-c)} + \frac{16}{9N} - \frac{151}{504}
\end{aligned}\right)\right]\\
&+\mathrm{Re}\left[\begin{aligned}
&2\left(\sum_{m=0}^{k} p'''_{0m}p'_{0(k-m)} - \sum_{m=0}^{k} p''_{0m}p''_{0(k-m)}\right)\left(-\frac{2}{15} - \frac{11c}{30(1-c)} + \frac{c^2}{3(1-c)^2}\right)\\
&-4\left(\sum_{m=0}^{k} p'''_{0m}p'_{0(k-m)} - \sum_{m=0}^{k} p''_{0m}p''_{0(k-m)}\right)\left(-\frac{1}{15} + \frac{19c}{30(1-c)} + \frac{3c^2}{20(1-c)^2}\right)
\end{aligned}\right]\\
&-\frac{\mathrm{Re}}{1-c}\left[-2\left(\sum_{m=0}^{k} p'''_{0m}p'_{0(k-m)} - \sum_{m=0}^{k} p''_{0m}p''_{0(k-m)}\right)\left(\begin{aligned}
&-\frac{15c}{2(1-c)} - \frac{c^2}{(1-c)^2} + \frac{8}{N}\\
&-\frac{4c}{3N(1-c)} - \frac{c}{3N} - \frac{13}{20}
\end{aligned}\right)\right]\\
&+\frac{L_p}{2}\left\{p_{0k} - p_{out} - \sigma H\left[C_{p1} - C_{out}\right]\right\},\\
&R_k^{C_{p0}}(z) = \sum_{m=0}^{k} p'_{0m}C'_{p0(k-m)}\\
&-\left[\frac{L_p}{2}\left\{p_{0k} - p_{out} - \sigma H\left[C_{p0k} - C_{out}\right]\right\}\right]\left[C_{p0k} - (1-\sigma)\bar{C}_{p0k}\right]\\
&+\frac{Ps}{2}\left(C_{p0k} - C_{out}\right).
\end{aligned}
\right\} \quad (91)
$$

The approximation for solution of k-order can be stated as

$$
\left.
\begin{aligned}
p_0(z) &= p_{00}(z) + \sum_{m=0}^{k} p_{0m}(z),\\
p_1(z) &= p_{10}(z) + \sum_{m=0}^{k} p_{1m}(z),\\
C_{p0}(z) &= C_{p00}(z) + \sum_{m=0}^{k} C_{p0m}(z).
\end{aligned}
\right\} \quad (92)
$$

The analytic solution of zeroth order pressure, first order pressure, zeroth order concentration up to second order approximations are obtained.

$$p_0(z) = \frac{3873}{500} - z + hz^2\left[-\frac{6617}{10000} - \frac{609\sigma}{625} + h\left[-\frac{6617}{10000} - \frac{609\sigma}{625}\right]\right] + hz^3\left[\frac{207}{5000} + \frac{3633\sigma}{10000} + h\left[\frac{259}{6250} - \frac{451\sigma}{5000} - \frac{2421\sigma^2}{5000} - \frac{3941\sigma^3}{5000}\right]\right] + h^2z^4\left[\frac{1047}{100000} + \frac{221\sigma}{10000} + \frac{1433\sigma^2}{10000} + \frac{249\sigma^3}{625}\right] + h^2z^5\left[-\frac{1}{2500} - \frac{\sigma}{1000} - \frac{7\sigma^2}{1250} - \frac{199\sigma^3}{2500}\right] \quad (93)$$

$$p_1(z) = \frac{213\mathrm{hz}^2}{10000} - \frac{9\mathrm{hz}^3}{1000} + \frac{3621\mathrm{hz}^3\sigma}{10000} + \frac{7c^4h^5z^4}{20000000N^2c^6} - \frac{0.00003003c^2h^5z^4\sigma^2}{N^3c^6} + \frac{4281h^2z^2}{100000c^3} - \frac{107ch^2z^2}{1250c^3} + \frac{427c^2h^2z^2}{10000c^3} + \frac{c^3h^2z^2}{c^3} + \frac{h^3z^2}{100000c^3} + \frac{21ch^3z^2}{10000c^3} - \frac{c^2h^3z^2}{1250c^3} - \frac{c^3h^3z^2}{1250c^3} + \frac{3h^2z^2}{20000N^2c^3} - \frac{27h^3z^2}{10000N^2c^3} + \frac{7ch^3z^2}{1250N^2c^3} - \frac{7c^2h^3z^2}{2500N^2c^3} - \frac{h^2z^2}{2500Nc^3} + \frac{ch^2z^2}{1250Nc^3} + \frac{17h^2z^3}{250c^3} - \frac{113ch^2z^3}{500c^3} - \frac{11ch^3z^3}{50000c^3} + \frac{c^2h^3z^3}{10000Nc^3} - \frac{17h^2z^4}{5000c^3} + \frac{13ch^2z^4}{1250c^3} - \frac{101c^2h^2z^4}{10000c^3} + \frac{17c^3h^2z^4}{5000c^3} + \frac{2.0579\times10^{-7}h^3z^4}{c^3} + \frac{0.0000085206ch^3z^4}{c^3} - \frac{0.0000033932c^2h^3z^4}{c^3} + \frac{9553ch^2z^2\sigma}{2500c^3} - \frac{487c^2h^2z^2\sigma}{250c^3} + \frac{3c^2h^2z^2\sigma}{5000N^2.c^3} - \frac{19h^3z^2\sigma}{5000N^2c^3} - \frac{21c^2h^2z^2\sigma}{10000Nc^3} + \frac{c^3h^2z^2\sigma}{5000Nc^3} + \frac{17h^3z^2\sigma}{10000Nc^3} + \frac{7883h^2z^3\sigma}{10000c^3} - \frac{11ch^2z^6\sigma}{10000c^3} - \frac{7c^2h^3z^2\sigma^2}{5000c^3} + \frac{127ch^3z^2\sigma^2}{10000N^2c^3} - \frac{99c^2h^2z^6\sigma^3}{2500c^3} + \frac{61c^3h^2z^6\sigma^3}{5000c^3} \quad (94)$$

$$C_{p0}(z) = z[-1 + h[\frac{12567}{10000} + \frac{13417\sigma}{10000} + \frac{21843\sigma^2}{10000}]] + 1.hz^2[-\frac{93}{2500} - \frac{993\sigma}{1250} - \frac{11039\sigma^2}{5000}] + hz^3[-\frac{217}{10000} + \frac{517\sigma}{10000} + \frac{3677\sigma^2}{5000}] \quad (95)$$

4 | RESULT AND DISCUSSION

The system of coupled differential equations (54)- (57), (62) and (63) is non-linear, the exact solution does not exist. Therefore, the series solution of above system is obtained through perturbation and then the values of pressure, average concentration, average axial velocity and radial velocity for fluid and particulate phase are determined using HAM. The graphical results of average concentration for fluid

and particulate, for greater values of reflection coefficient σ,where $\sigma$is the amount of solute particles passing through ultrafiltration, the average concentration for fluid and particulate phase within the sieve tube decreases (Figures:2 and Figure:8) and same is for pressure for fluid and particulate phase (the pressure within the sieve tube decreases (Figure:3 and Figure:9)). The results follow for the average axial velocity for fluid and particulate phase (Figure:4 and Figure:10) but opposite results are observed in radial velocity for fluid and particulate phase (Figure: 5 and Figure:11).The effect of reflection coefficient on radial velocity for both phases with the position of r, For greater reflection coefficient, the radial velocity for both phases within sieve tube decreases (Figure: 6 and Figure:12) but opposite results are observed in average axial velocity for both phases (Figure: 7 and Figure:13). Reflection coefficient σ varying between 0 to 0.3 (totally permeable membrane). Figures (2)- (5) and figures (8)- (11) are plotted for $r = 1$. Figures (6), (7), (12) and (13) are depicted for $z = 1$.

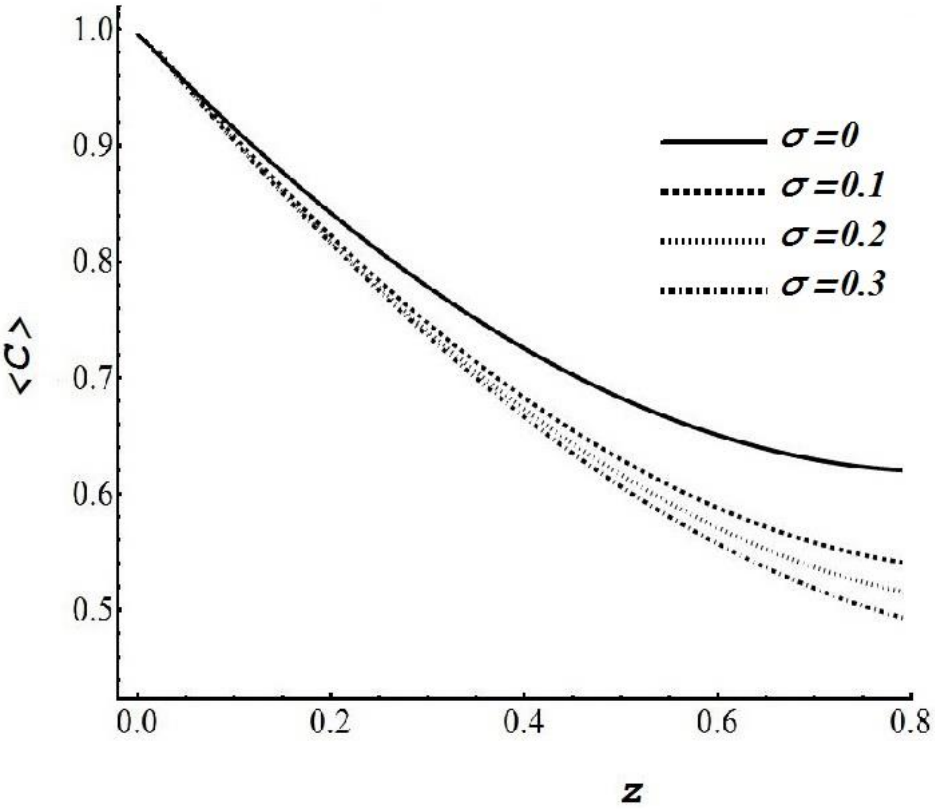


Figure 2. Effect of average concentration profile of fluid phase with position z with different reflection coefficients σ.

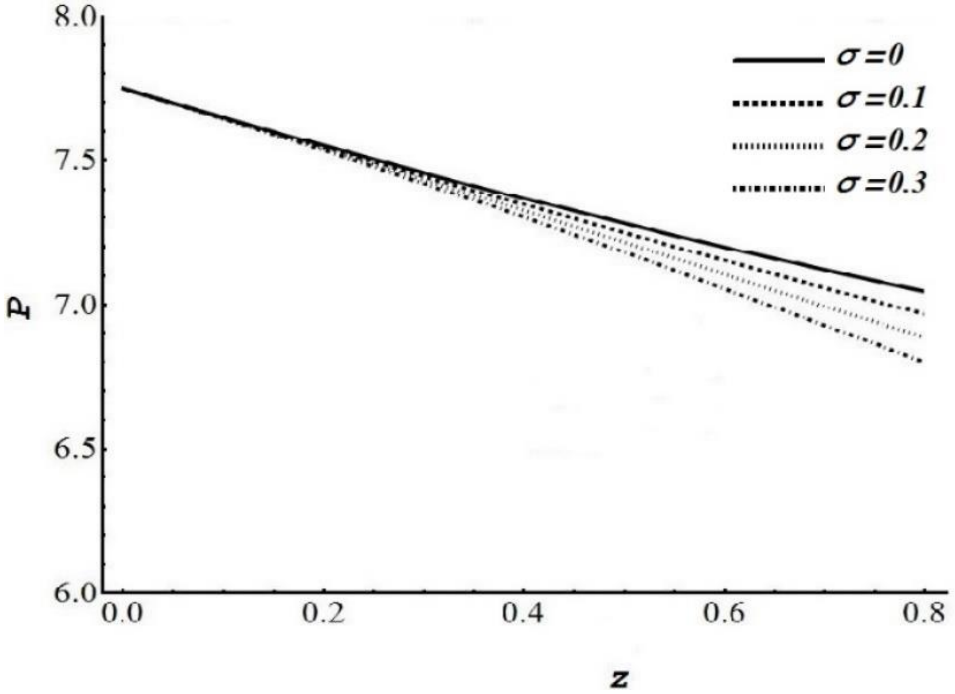


Figure 3. Effect of pressure profile of fluid phase with position z with different reflection coefficients σ.

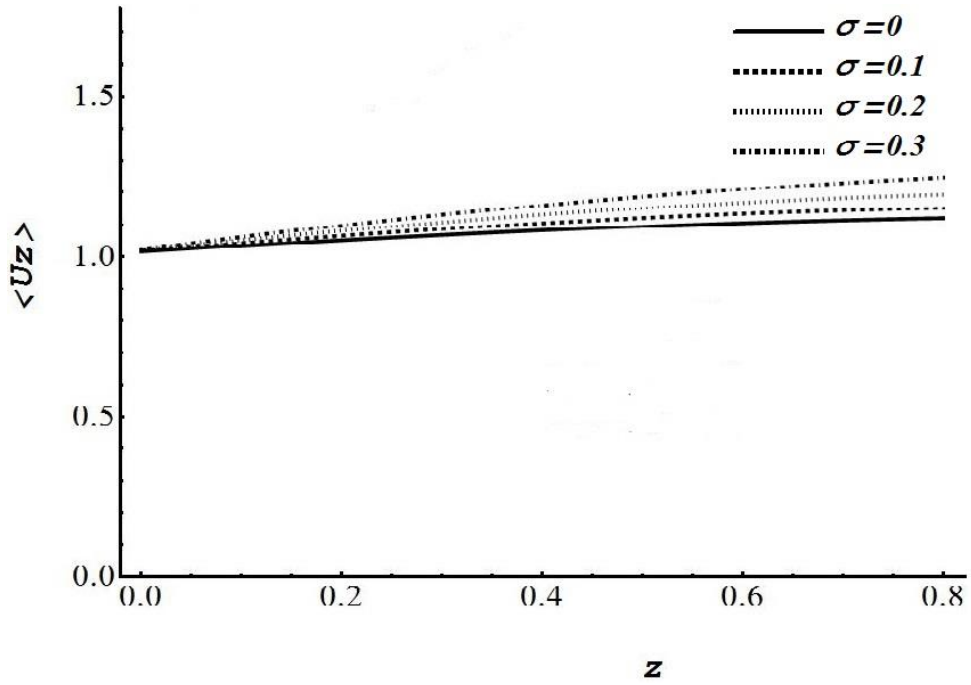


Figure 4. Effect of average axial velocity profile of fluid phase with position z with different reflection coefficients σ.

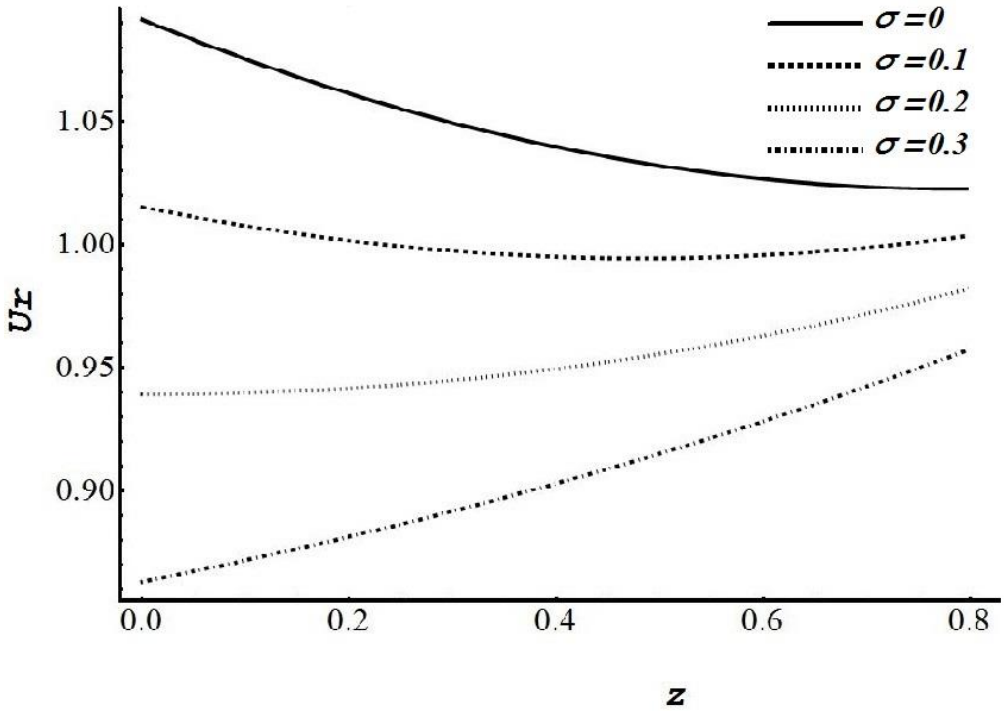


Figure 5. Effect of radial velocity profile of fluid phase with position *z* with different reflection coefficients σ.

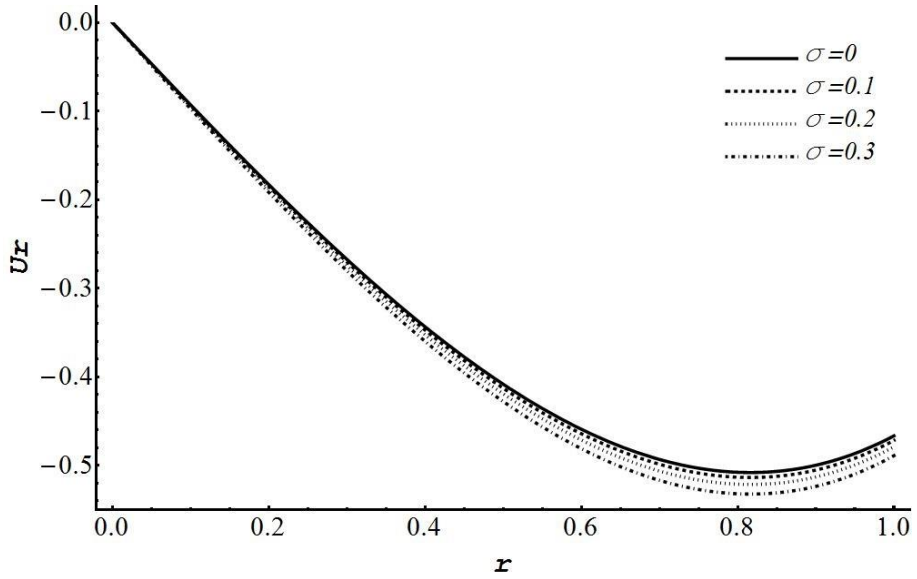


Figure 6. Effect of radial velocity profile of fluid phase with position *r* with different reflection coefficients σ.

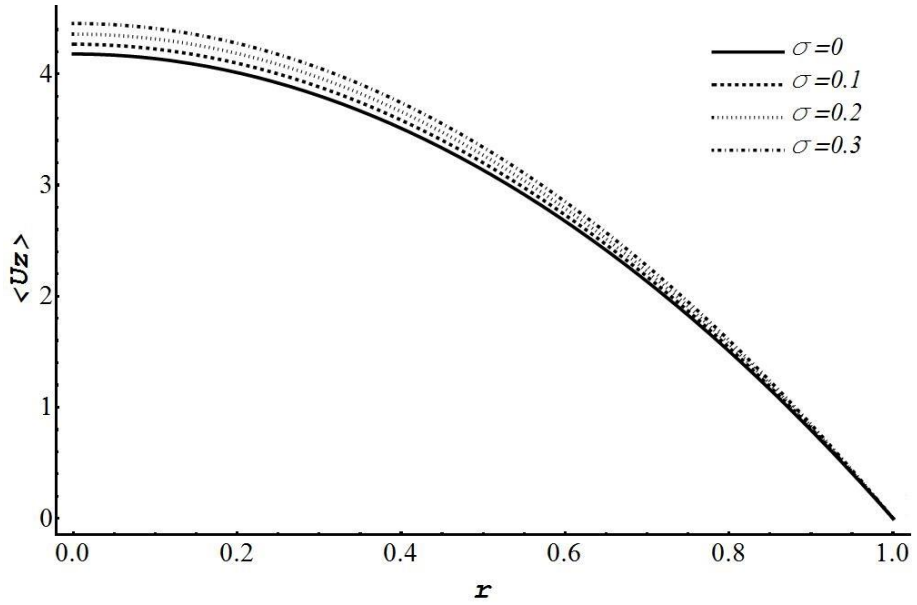


Figure 7. Effect of average axial velocity profile of fluid phase with position *r* with different reflection coefficients σ.

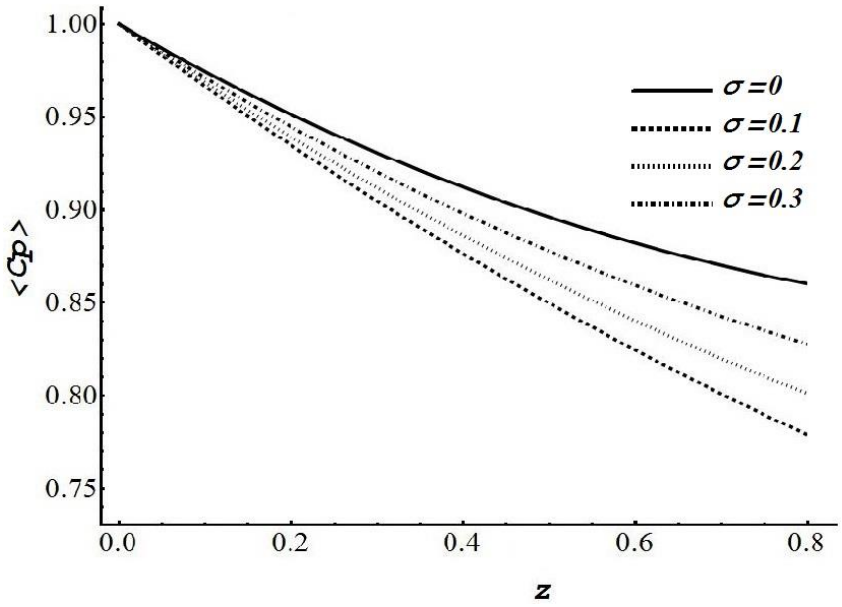


Figure 8. Effect of average concentration profile of particulate phase with position *z* with different reflection coefficients σ.

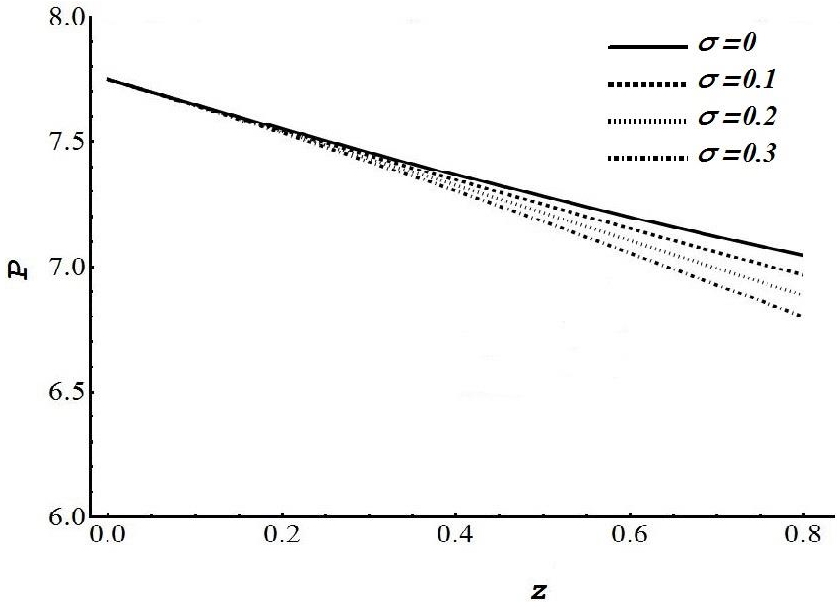
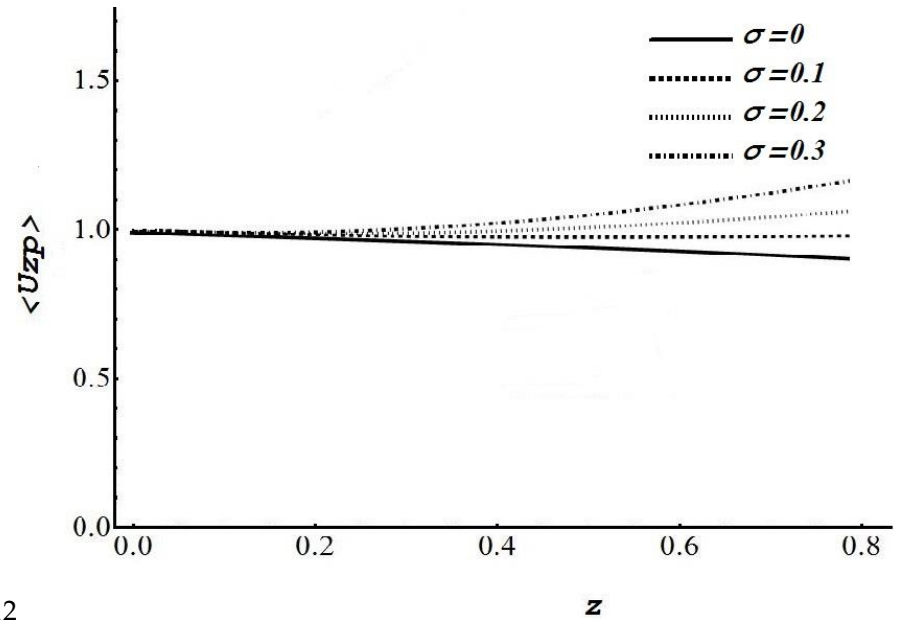


Figure 9. Effect of pressure profile of particulate phase with position z with different reflection coefficients σ.



12

Figure 10. Effect of average axial velocity profile of particulate phase with position z with different reflection coefficients σ.

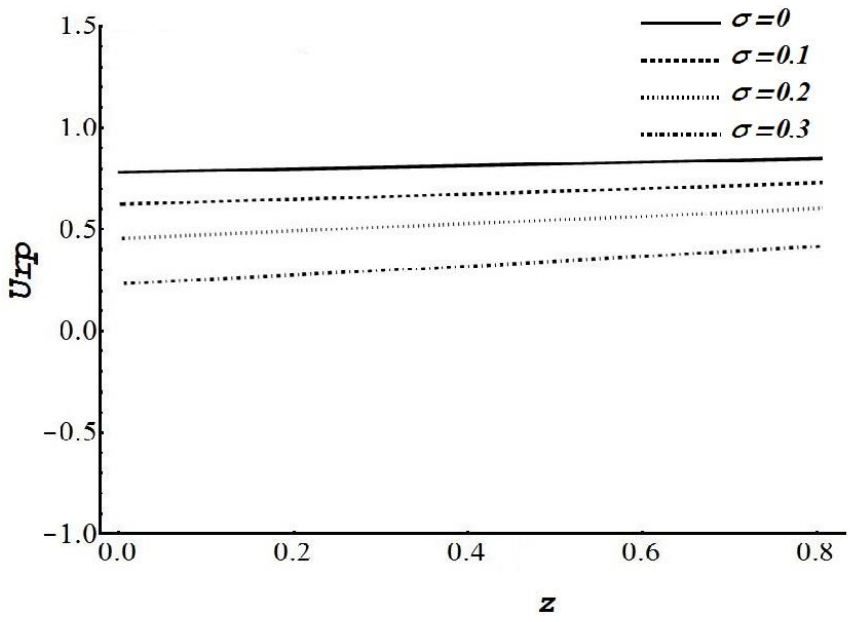


Figure 11. Effect of radial velocity profile of fluid phase with position z with different reflection coefficients σ.

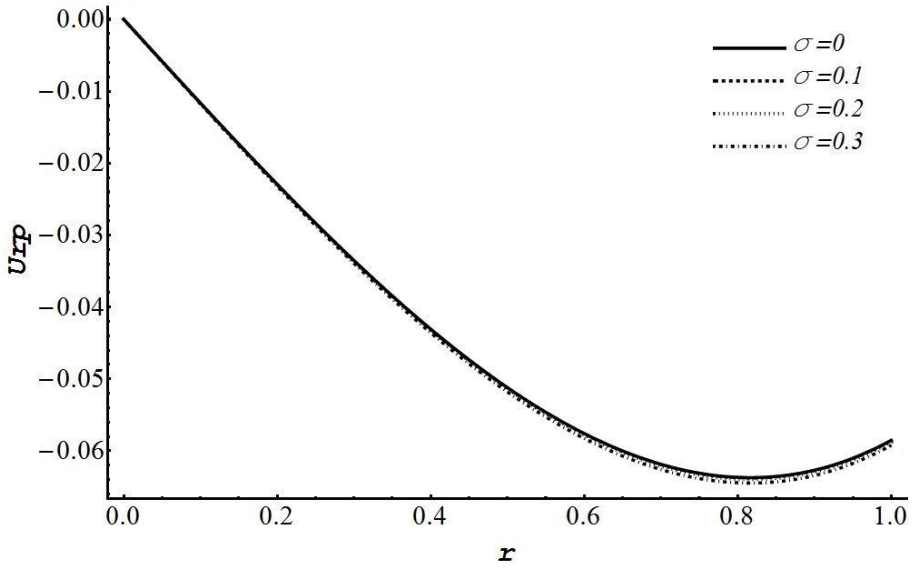


Figure 12. Effect of radial velocity profile of fluid phase with position *r* with different reflection coefficients σ.

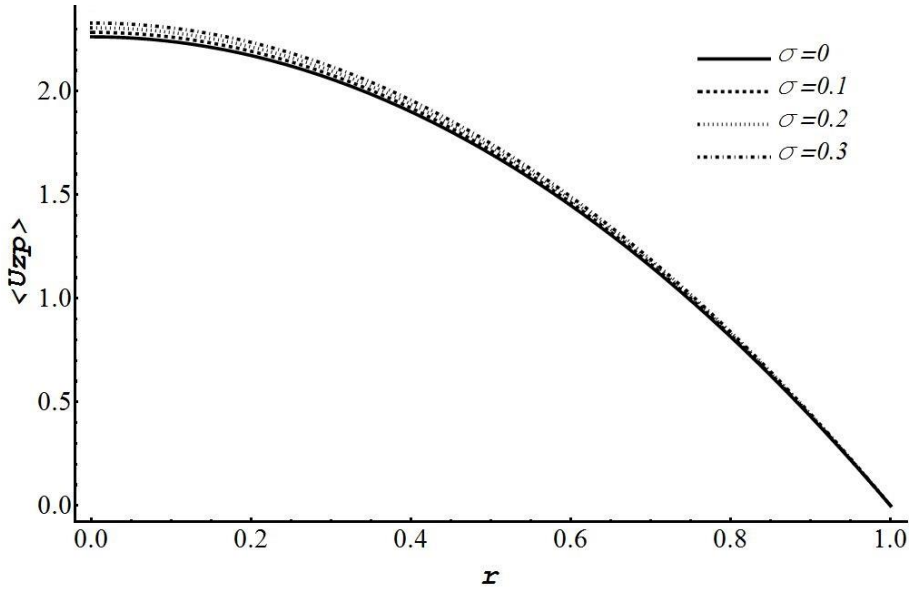


Figure 13. Effect of average axial velocity profile of particulate phase with position r with different reflection coefficients σ.

5 | CONCLUSION

In this study, equations are modelled using momentum and mass transfer. Perturbation is used to decompose PDEs two lower order systems of coupled equations, Then, HAM is applied to solve the problems. The effects of reflection coefficient on concentration, pressure, axial velocity and radial velocity for fluid and particles phases with the position z have been examined. The observed results are:

- For greater values of reflection coefficient, the average concentration for fluid and particles phases within sieve tube decreases.
- For higher values of reflection coefficient, the pressure for fluid and particles phases within sieve tube decreases.
- For greater reflection coefficient, the average axial velocity for fluid and particles phases within sieve tube increases.
- For greater values of reflection coefficient, the radial velocity for fluid and particles phases within sieve tube decreases.

The effects of reflection coefficient on axial velocity and radial velocity for fluid and particles phases with the position r have been examined. The observed results are

- For greater values of reflection coefficient, the average axial velocity for fluid and particles phases within sieve tube increases.
- For greater reflection coefficient, the radial velocity for fluid and particles phases within sieve tube decreases.

Acknowledgment: We are thankful to the editors and anonymous reviewers for their constructive suggestions.

Author Contributions: Conceptualization, A.R.; methodology, A.R. and M.B.A.; software, A.R. and M.B.A.; validation, A.R. and M.B.A.; investigation, A.R. and M.B.A.; data curation, A.R. and M.B.A.; writing—original draft preparation, A.R.; writing—review and editing, M.B.A. All authors have read and agreed to the

published version of the manuscript.

Declaration of Conflicting Interest: The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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Funding: The author(s) received no financial support for the research, authorship, and/or publication of this article.

Data Availability Statement: Data that supports the findings of this study are available on request from the corresponding author.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism was detected.

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