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Study of Second-Order Hankel Determinants for Convex Functions with Symmetric Points Related to Exponential Function

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ABSTRACT

This work aims to introduce and study a class of convex functions with symmetric points associated with exponential functions. The problems under consideration are the first four sharp coefficient bounds, the Fekete-Szegö functional, second Hankel determinant. It also includes similar investigations for inverse and logarithmic functions of convex functions with symmetric points associated with exponential functions.

Keywords: Convex functions with symmetric points; Coefficient problems; Zalcman functionals; Hankel determinant problems; Logarithmic Coefficient; Inverse Coefficient.
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1 | INTRODUCTION

When we investigate the complex-valued analytic and univalent functions, we observe that in making different estimations about these functions, there is a major role in studying coefficient problems. Talking about these problems, we have different types to study, some of which are coefficient-related bounds, necessary as well as sufficient conditions, covering results, both Hankel and Toeplitz determinants, and similarly many coefficient-related inequalities and conjectures. The well-known Hankel matrix was named after its introduction by Hermann Hankel. This structure is a square matrix which is symmetric too and particularly has the same entries in its skew diagonal. To study the above mentioned two important definitions in relation to each other, we form the Hankel matrix having the power series representation of analytic functions as its elements. Before going into details about this interesting relation, we give a brief collection of important definitions. If we consider an analytic function f, which is defined in the disk $\mathbb{D}=\{\upsilon\in\mathbb{C}:|\upsilon|<1\}$ and which also satisfies the conditions $f(0)=0$ and $f'(0)=1$, then the collection of such functions is denoted as A, and these functions can be written in the following power series/Taylor series form:

$$f(\upsilon)=\sum_{l=1}^{\infty}a_l\upsilon^l,\quad (a_1=1). \qquad (1.1)$$

Furthermore, recall that an analytic function is univalent in the region $\mathbb{D}$ if it takes no value more than once in $\mathbb{D}$. That is, f being univalent in $\mathbb{D}$ means mathematically that $f(\upsilon_1)=f(\upsilon_2)$ implies $\upsilon_1=\upsilon_2$ for $\upsilon_1,\upsilon_2\in\mathbb{D}$. Thus, by the symbol $\mathcal{S}$, we denote the class of univalent functions with series representation (1.1).

From Linear algebra, we know that the determinant associated with any square matrix is a significant number that highlights many algebraic and geometric characteristics of that matrix. The λ^{th} Hankel determinant for any element function of A which is associated with its matrix is defined based on the coefficients of the series given in (1.1), is given by:

$$\mathcal{H}_{\lambda,n}(f)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+\lambda-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+\lambda} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+\lambda-1} & a_{n+\lambda} & \cdots & a_{n+2\lambda-2} \end{vmatrix}.$$

Finding an upper bound function of this determinant is the main purpose, we study these determinants. When we variate the values of λ and n, we get different forms of the Hankel determinant $\mathcal{H}_{j,n}(f)$ given as:

$$\mathcal{H}_{2,1}(f)=\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}=a_1a_3-a_2^2,\qquad \mathcal{H}_{2,2}(f)=\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}=a_2a_4-a_3^2.$$

These determinants have their significance, for example, $\mathcal{H}_{2,1}$ is known as Fekete-Szegö functional [1] and $\mathcal{H}_{2,2}$ is another well-known and thoroughly researched, second Hankel determinant.

In history, we can trace back to the initial research taking place in 1966, by Pommerenke [2], [3] who developed the initial results and related the Hankel determinants with univalent functions as well as p-valent functions and starlike too. This was taken up by Noonan and Thomas [4] in 1976, who discussed the second Hankel determinants for p-valent functions. Then Noor contributed in this direction through multiple works published in 1983,1987,1992 [5], [6], [7] in which the author studied these special types of determinants for different types of functions including the close-to-convex functions, those functions which are defined with bounded boundary rotations and those called close to convex function of higher order. These researches can be found respectively in the published papers cited earlier. A similar kind of study was done by Ehrenborg [8] in 2000 but he did it for exponential polynomials. Layman followed his footsteps and got his work published in 2001 [9] where he presented a thorough study on the Hankel transform and explored some of its properties. Janteng et al. [10] 2007 extended the dimension of this research from the perspective of symmetric points of convex and starlike functions.In recent years, we have seen remarkable contributions by eminent researchers in this area. Realizing the importance of the use of Logarithmic functions in several applied sciences, Bilal et al. [11] in their article published last year, used the coefficients of these functions to work out the well-known problem of finding the bounds of the Hankel Determinant of order 3. They calculated these results for some sub-collections of starlike including sine functions. A couple of years ago, Hatun and Busra [12] did a tremendous job by presenting their research on the computation of the upper bound of the 4th order of this determinant and they did this for a special sub-collection of functions that comprised star-like but in particular those which were defined in perspective of modified sigmoid functions. In the same year, the well-known geometric function theorist Srivastava [13] along with his fellow researchers calculated similar result related to Hankel determinant of order 3, and they did it for entirely a new group of functions after defining them as "q-starlike function" that was defined using the "q-exponential functions". Their research opened new direction for the research output to come in future.Working on the lines of Zhang and Tang [14] studied similar problems for this determinant of order 4. They also presented their results for functions that were starlike but connected with sine functions. Various dimensions of research related to studying the connection between classes of complex valued functions and that of Hankel determinant could be observed evidently in these recent years. We find Kumar and Kumar [15] working differently and their research article published in 2021 brought the computation of some sharp and much refined results. Their results were related to the 3rd-order Hermitian-Toeplitz determinant and their bounds, both upper and lower. They did it for three different groups of starlike functions including strongly, Lemniscate, and lune. They carefully and vigilantly proved the connection of their presented research with a strong review of the literature. The research related to a collection of those functions which were connected with k-Fibonacci numbers and defined using q-derivative. This collection was thus named "q-starlike functions by the authors Shafiq et.al. [16]. The authors computed the upper bound of 3rd order Hankel for this new group of analytic functions. Highlighting the importance of geometrical properties of analytic functions, Murugusundaramoorthy et.al, [17] gave a powerful contribution by introducing a new sub collection of these functions using the well-known Carlson and Shaffer operator. Later on, while presenting their interesting results, they highlighted the fact that their considered functions were related with a shell shaped function. The variety of results that they proved in their article included the bounds of power series expansion's 4 coefficients as well as the related results for Fekete-Szegö and for 2nd order Hankel determinant for those functions that were elements of their newly defined group.The readers who are interested to know the initial research appearing before these fantastic results can go through Mahmood et.al [18] and Srivastava et.al. [19]. These articles were published in year 2019. The former of them worked on computation of bounds of 3rd order Hankel for those functions which were defined in perspective of the famous operator known as "Ruscheweyh-type q-derivative". The latter group calculated these results for a new group of functions which was introduced by them using the lemniscate of Bernoulli. They backed up their results with strong references and by giving proofs of some more relevant results. Inspired by this earlier work, 3 years later, some interesting results were proved by Raza and his team [20] which was a much more generalized version. They presented the results related to Hankel and the bounds of coefficients. Their results were meant for a sub-collection of starlike functions but those which were particularly relevant to Booth lemniscate. They also computed the 1st four coefficient bounds which were mathematically refined (sharp) also 2nd and 3rd ordered Hankel determinants as well as they proved the Zalcman conjecture for their newly defined collection of functions.

For some more interesting results, one can also see [21], [22], [22], [23], [24], [25], [26] and the references that are quoted in these articles.

In 1959, Sakaguchi [27] generalized class $\mathcal{S}^* = \left\{f \in \mathcal{S}: \Re \frac{vf'(v)}{f(v)} > 0 \qquad (v \in \mathbb{D})\right\}$ of starlike functions by introducing the class $\mathcal{S}_s^*$ of starlike functions associated with symmetric points. In 1977, Das and Singh [28] used this concept to develop the family $\mathcal{K}_s$ of convex functions with symmetric points. In both of these papers, the family $\mathcal{S}_s^*$ and $\mathcal{K}_s$ are defined as follow

$$\mathcal{S}_s^* = \left\{f \in \mathcal{S}: \Re \frac{2vf'(v)}{f(v) - f(-v)} > 0 \qquad (v \in \mathbb{D})\right\},$$

$$\mathcal{K}_s = \left\{f \in \mathcal{S}: \Re \frac{2\big(vf'(v)\big)'}{\big(f(v) - f(-v)\big)'} > 0 \qquad (v \in \mathbb{D})\right\}.$$

Sakaguchi stated that $\mathcal{S}_s^*$ a subclass of the class $\mathcal{C}$ the class of close to convex function contains the class convex and odd starlike function about origin. After that, many researchers present several new univalent function subclasses about symmetric points and discuss coefficient-type results; see a few of them in [29], [30], [31], [32], [33].

Now, due to the above facts, we define the family $\mathcal{SK}_{\exp}$ by the following representation:

$$\mathcal{SK}_{\exp} = \left\{f \in \mathcal{S}: \frac{2\big(vf'(v)\big)'}{\big(f(v)-f(-v)\big)'} \prec e^v \qquad (v \in \mathbb{D})\right\}. \qquad (1.2)$$

In this paper, we investigate the sharp estimates of coefficients, Fekete-Szegö, Zalcman inequalities, and sharp bound of a second-order determinant for the class $\mathcal{SK}_{\exp}$ of convex function for symmetric points linked with exponential function. Further, we also investigate the inverse and logarithmic coefficients for the same class.

2 | SET OF LEMMAS

Let also $\mathcal{L}_0$ be the family of Schwarz functions, i.e., holomorphic functions $w: \mathbb{D} \to \mathbb{D}$, $w(0) = 0$. We can write the function $w \in \mathcal{L}_0$ as a power series

$w(\upsilon)=\sum_{n=1}^{\infty}\xi_n\upsilon^n$. (2.1)

Lemma 2.1 [34] Let $w(\upsilon)=\sum_{n=1}^{\infty}\xi_n\upsilon^n$ be a Schwarz function. Then for any real numbers σ and ς with $(\sigma,\varsigma)\in\mathbb{D}_1\cup\mathbb{D}_2$, we have the following sharp estimate given by

$$|\xi_3+\sigma\xi_1\xi_2+\varsigma\xi_1^3|\leq 1.$$

Where

$$\mathbb{D}_1=\left\{|\sigma|\leq\frac{1}{2},-1\leq\varsigma\leq 1\right\},$$

$$\mathbb{D}_2=\left\{\frac{1}{2}\leq|\sigma|\leq 2,\frac{4}{27}(1+|\sigma|)^3-(1+|\sigma|)\leq\varsigma\leq 1\right\}.$$

Lemma 2.2 [35] If $w\in\mathcal{L}_0$ is in the form of (2.1), then

$$|\xi_2|\leq 1-|\xi_1|^2, \quad (2.2)$$

$$|\xi_n|\leq 1, n\geq 1. \quad (2.3)$$

Moreover, the inequality of (2.2) can be improved in the manner of

$$|\xi_2+\eta\xi_1^2|\leq\max\{1,|\eta|\},\quad \eta\in\mathbb{C}. \quad (2.4)$$

Lemma 2.3 [36] Let $w(\upsilon)=\xi_1\upsilon+\xi_2\upsilon^2+\ldots$ be a Schwarz function. Then

$$|\xi_3|\leq 1-|\xi_1|^2-\frac{|\xi_2|^2}{1+|\xi_1|}, \quad (2.5)$$

$$|\xi_4|\leq 1-|\xi_1|^2-|\xi_2|^2. \quad (2.6)$$

Lemma 2.4 [37] Let $w(\upsilon)=\xi_1\upsilon+\xi_2\upsilon^2+\ldots$ be a Schwarz function. Then

$$|\xi_1\xi_3-\xi_2^2|\leq 1-|\xi_1|^2.$$

Lemma 2.5 [38] Let $w(\upsilon)=\xi_1\upsilon+\xi_2\upsilon^2+\ldots$ be a Schwarz function and $\gamma\in\mathbb{C}$. Then

$$|\xi_4+2\xi_1\xi_3+\eta\xi_2^2+(1+2\eta)\xi_1^2\xi_2+\eta\xi_1^4|\leq\max\{1,|\eta|\}, \quad (2.7)$$

and

$$|\xi_4+(1+\eta)\xi_1\xi_3+\xi_2^2+(1+2\eta)\xi_1^2\xi_2+\eta\xi_1^4|\leq\max\{1,|\eta|\}. \quad (2.8)$$

There is much greater potential in the method that Efraimidis used to prove his lemma. Inequalities involving the fifth coefficient of $w\in\mathcal{L}_0$ can be obtained using this approach [39].

3 | COEFICIENT BOUNDS FOR $f\in\mathcal{SK}_{\exp}$

We first start with the coefficient bounds of $f\in\mathcal{SK}_{\exp}$.

Theorem 3.1 Let $f\in\mathcal{SK}_{exp}$. Then

$$|a_2|\leq\frac{1}{4},$$

$$|a_3|\leq\frac{1}{6},$$

$$|a_4|\leq\frac{1}{16},$$

$$|a_5|\leq\frac{1}{20}.$$

These all bounds are sharp.

Proof. Assume that $f \in \mathcal{SK}_{\exp}$. From the definition, we know there exists a Schwarz function w such that

$$\frac{2\left(vf'(v)\right)'}{\left(f(v)-f(-v)\right)'} = e^{w(v)}. \qquad (3.1)$$

Utilizing (1.1), we get

$$\frac{2\left(vf'(v)\right)'}{\left(f(v)-f(-v)\right)'} := 1 + 4a_2v + 6a_3v^2 + (-12a_2a_3 + 16a_4)v^3 + (-18a_3^2 + 20a_5)v^4 + \cdots. \qquad (3.2)$$

Let

$$w(v) = \xi_1 v + \xi_2 v^2 + \xi_3 v^3 + \xi_4 v^4 + \cdots. \qquad (3.3)$$

After easy computation and the series expansion of (3.3), we obtain

$$e^{w(v)} = 1 + \xi_1 v + \left(\xi_2 + \frac{1}{2}\xi_1^2\right)v^2 + \left(\xi_3 + \xi_1\xi_2 + \frac{1}{6}\xi_1^3\right)v^3$$
$$+ \left(\xi_4 + \xi_1\xi_3 + \frac{1}{2}\xi_2^2 + \frac{1}{24}\xi_1^4 + \frac{1}{2}\xi_1^2\xi_2\right)v^4 + \cdots. \qquad (3.4)$$

Now by comparing (3.2) and (3.4), we have

$$a_2 = \frac{1}{4}\xi_1, \qquad (3.5)$$

$$a_3 = \frac{1}{6}\xi_2 + \frac{1}{12}\xi_1^2, \qquad (3.6)$$

$$a_4 = \frac{1}{16}\xi_3 + \frac{5}{192}\xi_1^3 + \frac{3}{32}\xi_1\xi_2, \qquad (3.7)$$

$$a_5 = \frac{1}{20}\xi_2^2 + \frac{1}{20}\xi_4 + \frac{1}{20}\xi_1^2\xi_2 + \frac{1}{20}\xi_1\xi_3 + \frac{1}{120}\xi_1^4. \qquad (3.8)$$

The bounds of a_2 and a_3 are clear. For a_4 rearranging, we get

$$|a_4| = \frac{1}{16}\left|\xi_3 + \frac{3}{2}\xi_1\xi_2 + \frac{5}{12}\xi_1^3\right|.$$

By applying triangle inequality and **Lemma 2.1** with $\sigma = \frac{3}{2}$ and $\varsigma = \frac{5}{12}$, we obtain

$$|a_4| \le \frac{1}{16}.$$

Reshuffling (3.8), we have

$$20|a_5| = \left|\frac{1}{2}(\xi_4 + 2\xi_1\xi_3 + \xi_2^2 + 3\xi_1^2\xi_2 + \xi_1^4) + \frac{1}{2}\left(\xi_4 + \xi_2^2 - \frac{2}{3}\xi_1^4 - \xi_1^2\xi_2\right)\right|. \qquad (3.9)$$

From (2.8) with $\eta = 1$, the first component is bounded by 1/2. By **Lemma 2.3**, the second component can be estimated as follows.

$$\frac{1}{2}\left|\xi_2^2 + \xi_4 - \frac{2}{3}\xi_1^4 - \xi_1^2\xi_2\right| \le \frac{1}{2}\left[1 - |\xi_1|^2 - |\xi_2|^2 + |\xi_2|^2 + \frac{2}{3}|\xi_1|^4 + |\xi_1|^2(1 - |\xi_1|^2)\right]$$
$$= \frac{1}{2} - \frac{1}{6}|\xi_1|^4 \le \frac{1}{2}.$$

Adding the estimates of both components of (3.9), we get

$$|a_5| \le \frac{1}{20}.$$

The bounds on the estimation of $|a_2|$, $|a_3|$, $|a_4|$ and $|a_5|$ are sharp with the extremal functions given respectively by

$$\frac{2\left(vf'(v)\right)'}{\left(f(v)-f(-v)\right)'}=e^{(v)}=1+v+\frac{1}{2}v^2+\ldots, \tag{3.10}$$

$$\frac{2\left(vf'(v)\right)'}{\left(f(v)-f(-v)\right)'}=e^{(v^2)}=1+v^2+\frac{1}{2}v^4+\ldots, \tag{3.11}$$

$$\frac{2\left(vf'(v)\right)'}{\left(f(v)-f(-v)\right)'}=e^{(v^3)}=1+v^3+\frac{1}{2}v^6+\ldots, \tag{3.12}$$

$$\frac{2\left(vf'(v)\right)'}{\left(f(v)-f(-v)\right)'}=e^{(v^4)}=1+v^4+\frac{1}{2}v^8+\ldots \tag{3.13}$$

Theorem 3.2 Let $f\in\mathcal{SK}_{exp}$. Then

$$|a_3-\eta a_2^2|\leq \max\left\{\frac{1}{6},\left|\frac{4-3\eta}{48}\right|\right\}.$$

This result is sharp.

Proof. From (3.5) and (3.6), we achieve

$$|a_3-\eta a_2^2|=\frac{1}{6}\left|\xi_2+\frac{1}{2}\xi_1^2-\frac{3}{8}\eta\xi_1^2\right|,$$
$$=\frac{1}{6}\left|\xi_2+\left(\frac{4-3\eta}{8}\right)\xi_1^2\right|.$$

Implementation of **Lemma 2.2** and triangle inequality, we get

$$|a_3-\eta a_2^2|\leq \max\left\{\frac{1}{6},\left|\frac{4-3\eta}{48}\right|\right\}.$$

Putting $\eta=1$, we get the following result.

Corollary 3.3 If $f\in\mathcal{SK}_{exp}$ is of the form (1.1), then

$$|a_3-a_2^2|\leq\frac{1}{6}.$$

Equality holds for the extremal function given by (3.11).

Theorem 3.4 Assume that $f\in\mathcal{SK}_{exp}$ be the form of (1.1), then

$$|a_4-a_2a_3|\leq\frac{1}{16}.$$

This result is sharp for the extremal function provided by (3.12).

Proof. From (3.5), (3.6) and (3.7), we obtain

$$|a_4-a_2a_3|=\frac{1}{16}\left|\xi_3+\frac{5}{6}\xi_1\xi_2+\frac{1}{12}\xi_1^3\right|,$$

So taking $\sigma=\frac{5}{6}$ and $\varsigma=\frac{1}{12}$ in Lemma 2.1 yields

$$|a_4-a_2a_3|\leq\frac{1}{16}.$$

Which is the required proof.

Theorem 3.5 Let $f\in\mathcal{SK}_{exp}$. Then

$$\left|\mathcal{H}_{2,2}\right|\leq\frac{1}{36}.$$

Equality is hold for the extremal function given by (3.11).

Proof. From (3.5), (3.6) and (3.7), we have

$$|a_2a_4 - a_3^2| = \frac{1}{36}\left|\xi_2^2 + \frac{1}{64}\xi_1^4 - \frac{9}{16}\xi_1\xi_3 + \frac{5}{32}\xi_1^2\xi_2\right|$$

$$= \frac{1}{36}\left|\frac{1}{2}(\xi_2^2 - \xi_1\xi_3) + \frac{1}{2}\left(\frac{1}{32}\xi_1^4 - \frac{1}{8}\xi_1\xi_3 + \frac{5}{16}\xi_1^2\xi_2 + \xi_2^2\right)\right|$$

$$\leq \frac{1}{72}|\xi_2^2 - \xi_1\xi_3| + \frac{1}{72}\left|\frac{1}{32}\xi_1^4 - \frac{1}{8}\xi_1\xi_3 + \frac{5}{16}\xi_1^2\xi_2 + \xi_2^2\right|$$

$$= \frac{1}{72}M_1 + \frac{1}{72}M_2,$$

where

$$M_1 = |\xi_2^2 - \xi_1\xi_3|$$

and

$$M_2 = \left|\frac{1}{32}\xi_1^4 - \frac{1}{8}\xi_1\xi_3 + \frac{5}{16}\xi_1^2\xi_2 + \xi_2^2\right|.$$

Using **Lemma 2.4**, we get $M_1 \leq 1$. For M_2 using triangle inequality and Lemma 2.3, we have

$$|M_2| \leq \frac{1}{32}|\xi_1|^4 + \frac{1}{8}|\xi_1|\left(1 - |\xi_1|^2 - \frac{|\xi_2|^2}{1 + |\xi_1|}\right) + \frac{5}{16}|\xi_1|^2|\xi_2| + |\xi_2|^2,$$

$$\leq \frac{1}{32}|\xi_1|^4 + \frac{1}{8}|\xi_1| - \frac{1}{8}|\xi_1|^3 - \frac{|\xi_1||\xi_2|^2}{8(1 + |\xi_1|)} + \frac{5}{16}|\xi_1|^2|\xi_2| + |\xi_2|^2,$$

$$\leq \frac{1}{32}|\xi_1|^4 + \frac{1}{8}|\xi_1| - \frac{1}{8}|\xi_1|^3 + |\xi_2|^2\left(1 - \frac{|\xi_1|}{8(1+|\xi_1|)}\right) + \frac{5}{16}|\xi_1|^2|\xi_2|. \qquad (3.14)$$

Since $\left(1 - \frac{|\xi_1|}{8(1+|\xi_1|)}\right) > 0$. Therefore, we can put $|\xi_2| \leq 1 - |\xi_1|^2$ in (3.14), we have

$$|M_2| \leq \frac{1}{32}|\xi_1|^4 + \frac{1}{8}|\xi_1| - \frac{1}{8}|\xi_1|^3 + (1 - |\xi_1|^2)^2\left(1 - \frac{|\xi_1|}{8(1 + |\xi_1|)}\right) + \frac{5}{16}|\xi_1|^2(1 - |\xi_1|^2).$$

Let us put $|\xi_1| = x$ and $x \in (0,1]$, we get

$$|M_2| \leq 1 - \frac{25}{16}x^2 + \frac{19}{32}x^4 = \mathrm{F}(x).$$

As $\mathrm{F}'(x) \leqq 0$, $\mathrm{F}(x)$ is decreasing function of x, so that it gives the maximum value at $x = 0$

$|M_2| \leq 1.$

Hence

$\left|\mathcal{H}_{2,2}\right| \leq \frac{1}{72}M_1 + \frac{1}{72}M_2 \leq \frac{1}{36}.$

4 | LOGARITHMIC COEFFICIENT FOR $\mathcal{SK}_{\exp}$

The logarithmic coefficients of a given function f, represented by $\gamma_n = \gamma_n(f)$, are described as

$$\frac{1}{2}\log\left(\frac{f(v)}{v}\right) = \sum_{n=1}^{\infty} \gamma_n v^n. \qquad (4.1)$$

The concept of extrapolating the Hankel determinant with logarithmic coefficients as an entry appears obvious. In [40], [41], [42], Kowalczyk et al. first introduced the Hankel determinant using logarithmic coefficients as the element, we have:

$$\mathcal{H}_{q,n}(F_f/2) := \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2q-2} \end{vmatrix}. \qquad (4.2)$$

In particular, it is noted that

$$\mathcal{H}_{2,1}(F_f/2) = \begin{vmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 \end{vmatrix} = |\gamma_1\gamma_3 - \gamma_2^2|.$$

For more studies of logarithmic coefficients, see the articles [43], [44].

If f is given by (1.1), then its logarithmic coefficients are given as follows

$$\gamma_1 = \frac{1}{2}a_2 \qquad (4.3)$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right) \qquad (4.4)$$

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right). \qquad (4.5)$$

Theorem 4.1 Let $f \in \mathcal{SK}_{exp}$. Then

$$|\gamma_1| \le \frac{1}{8},$$

$$|\gamma_2| \le \frac{1}{12},$$

$$|\gamma_3| \le \frac{1}{32}.$$

These all bounds are sharp.

Proof. Applying (3.5), (3.6), (3.7) in (4.3), (4.4), and (4.5), we get

$$\gamma_1 = \frac{1}{8}\xi_1, \qquad (4.6)$$

$$\gamma_2 = \frac{1}{12}\xi_2 + \frac{5}{192}\xi_1^2, \qquad (4.7)$$

$$\gamma_3 = \frac{1}{32}\xi_3 + \frac{5}{192}\xi_1\xi_2 + \frac{1}{192}\xi_1^3. \qquad (4.8)$$

The bounds of γ_1 and γ_2 are clear. For γ_3 rearranging, we get

$$|\gamma_3| = \frac{1}{32}\left|\xi_3 + \frac{5}{6}\xi_1\xi_2 + \frac{1}{6}\xi_1^3\right|.$$

Applying triangle inequality and **Lemma 2.1** with $\sigma = \frac{5}{6}$ and $\varsigma = \frac{1}{6}$, we obtain

$$|\gamma_3| \le \frac{1}{32}.$$

Equalities hold for the function given (3.10), (3.11), (3.12) and using (4.3), (4.4), (4.5).

Theorem 4.2 If f $\in \mathcal{SK}_{\text{exp}}$ is of the form (1.1), then

$$|\gamma_2 - \eta\gamma_1^2| \le \max\left\{\frac{1}{12}, \left|\frac{5-3\eta}{192}\right|\right\}.$$

This result is sharp.

Proof. From (4.6) and (4.7), we get

$$|\gamma_2 - \eta\gamma_1^2| = \frac{1}{12}\left|\xi_2 + \frac{5}{16}\xi_1^2 - \frac{3\eta}{16}\xi_1^2\right|,$$

$$= \frac{1}{12}\left|\xi_2 + \left(\frac{5-3\eta}{16}\right)\xi_1^2\right|.$$

Implementation of **Lemma 2.2** and triangle, we get.

$$|\gamma_2 - \eta\gamma_1^2| \leq \max\left\{\frac{1}{12}, \left|\frac{5-3\eta}{192}\right|\right\}.$$

Putting $\eta = 1$, we get the bellow corollary.

Corollary 4.3 If $f \in \mathcal{SK}_{exp}$ is of the form (1.1), then

$$|\gamma_2 - \gamma_1^2| \leq \frac{1}{12}.$$

Equality is determined by using (4.3), (4.4) and (3.11).

Theorem 4.4 Let $f \in \mathcal{SK}_{exp}$. Then

$$|\gamma_3 - \gamma_1\gamma_2| \leq \frac{1}{32}.$$

This result is sharp. Equality is determined by using (4.3), (4.4), (4.5) and (3.12).

Proof. From (4.6), (4.7) and (4.8), we get

$$|\gamma_3 - \gamma_1\gamma_2| = \frac{1}{32}\left|\xi_3 + \frac{1}{2}\xi_1\xi_2 + \frac{1}{16}\xi_1^3\right|,$$

so, taking $\sigma = \frac{1}{2}$ and $\varsigma = \frac{1}{16}$ in **Lemma 2.1** yields

$$|\gamma_3 - \gamma_1\gamma_2| \leq \frac{1}{32}.$$

Which completes the proof.

Theorem 4.5 If $f \in \mathcal{SK}_{exp}$ is of the form (1.1), then

$$\left|\mathcal{H}_{2,1}(F_f/2)\right| = |\gamma_1\gamma_3 - \gamma_2^2| \leq \frac{1}{144}.$$

This result is sharp. Equality is held by using (4.3), (4.4), (4.5) and (3.11).

Proof. From (4.6), (4.7) and (4.8), we have

$$|\gamma_1\gamma_3 - \gamma_2^2| = \frac{1}{144}\left|\xi_2^2 + \frac{1}{256}\xi_1^4 - \frac{9}{16}\xi_1\xi_3 + \frac{5}{32}\xi_1^2\xi_2\right|$$

$$= \frac{1}{144}\left|\frac{1}{2}(\xi_2^2 - \xi_1\xi_3) + \frac{1}{2}\left(\frac{1}{128}\xi_1^4 - \frac{1}{8}\xi_1\xi_3 + \frac{5}{16}\xi_1^2\xi_2 + \xi_2^2\right)\right|$$

$$\leq \frac{1}{288}|\xi_2^2 - \xi_1\xi_3| + \frac{1}{288}\left|\frac{1}{128}\xi_1^4 - \frac{1}{8}\xi_1\xi_3 + \frac{5}{16}\xi_1^2\xi_2 + \xi_2^2\right|$$

$$= \frac{1}{288}N_1 + \frac{1}{288}N_2,$$

where

$$N_1 = |\xi_2^2 - \xi_1\xi_3|$$

and

$$N_2 = \left|\frac{1}{128}\xi_1^4 - \frac{1}{8}\xi_1\xi_3 + \frac{5}{16}\xi_1^2\xi_2 + \xi_2^2\right|$$

Using **Lemma 2.4**, we achieve $N_1 \leq 1$. For N_2 using triangle inequality and **Lemma 2.3**, we have

$$|N_2| \leq \frac{1}{128}|\xi_1|^4 + \frac{1}{8}|\xi_1|\left(1 - |\xi_1|^2 - \frac{|\xi_2|^2}{1 + |\xi_1|}\right) + \frac{5}{16}|\xi_1|^2|\xi_2| + |\xi_2|^2,$$

$$\leq \frac{1}{128}|\xi_1|^4 + \frac{1}{8}|\xi_1| - \frac{1}{8}|\xi_1|^3 - \frac{|\xi_1||\xi_2|^2}{8(1 + |\xi_1|)} + \frac{5}{16}|\xi_1|^2|\xi_2| + |\xi_2|^2,$$

$$\leq \frac{1}{128}|\xi_1|^4 + \frac{1}{8}|\xi_1| - \frac{1}{8}|\xi_1|^3 + |\xi_2|^2\left(1 - \frac{|\xi_1|}{8(1+|\xi_1|)}\right) + \frac{5}{16}|\xi_1|^2|\xi_2|. \qquad (4.9)$$

Since $\left(1 - \frac{|\xi_1|}{8(1+|\xi_1|)}\right) > 0$. Therefore, we can put $|\xi_2| \leq 1 - |\xi_1|^2$ in (4.9), we have

$$|N_2| \leq \frac{1}{128}|\xi_1|^4 + \frac{1}{8}|\xi_1| - \frac{1}{8}|\xi_1|^3 + (1 - |\xi_1|^2)^2\left(1 - \frac{|\xi_1|}{8(1 + |\xi_1|)}\right) + \frac{5}{16}|\xi_1|^2(1 - |\xi_1|^2).$$

After elementary calculus of maxima and minima, we get

$$|N_2| \leq 1.$$

Hence

$$\left|\mathcal{H}_{2,1}(F_f/2)\right| \leq \frac{1}{288}N_1 + \frac{1}{288}N_2 \leq \frac{1}{144}.$$

Hence this is our required proof.

5 | INVERSE COEFFICIENT FOR $\mathcal{SK}_{\mathbf{exp}}$

The renowned Köebe 1/4-theorem ensures that for each univalent function f defined in $\mathbb{D}$, its inverse f^{-1} exists at least on a disc of radius 1/4 with Taylor's series of the form representation

$$f^{-1}(\xi) = \xi + \sum_{n=2}^{\infty} A_n \xi^n, \quad \left(|\xi| < \tfrac{1}{4}\right). \qquad (5.1)$$

Using the representation $f(f^{-1}(\xi)) = \xi$, we get

$$A_2 = -a_2 \quad (5.2)$$

$$A_3 = -a_3 + 2a_2^2 \quad (5.3)$$

$$A_4 = -a_4 + 5a_2a_3 - 5a_2^3. \quad (5.4)$$

Understanding the geometric behavior of the inverse function has attracted a lot of attention from scholars recently. For example, Krzyz et al. [45] calculated the upper bounds of the initial coefficient contained in the inverse function f^{-1} when $f \in \mathcal{S}^*(\beta)$ with $0 \leq \beta \leq 1$. Further, Ali [46] find the sharp bounds of the first four initial coefficients for the family $\mathcal{SS}^*(\zeta)$ $(0 < \zeta \leq 1)$ of a strongly starlike function as well as the sharp estimate of the Fekete–Szegö coefficient functional of the inverse function. For study of inverse coefficients, see the articles [47], [48], [49], [50].

Theorem 5.1 Let $f \in \mathcal{SK}_{exp}$. Then

$$|A_2| \leq \frac{1}{4},$$

$$|A_3| \leq \frac{1}{6}.$$

These bounds are sharp.

Proof. Applying (3.5), (3.6), (3.7) in (5.2), (5.3), and (5.4), we get

$$A_2 = -\frac{1}{4}\xi_1 \quad (5.5)$$

$$A_3 = \frac{1}{24}\xi_1^2 - \frac{1}{6}\xi_2 \qquad (5.6)$$

$$A_4 = \frac{11}{96}\xi_1\xi_2 - \frac{1}{16}\xi_3. (5.7)$$

From (5.5), (5.6) using **Lemma 2.2** and a triangle, we get

$$|A_2| \leq \frac{1}{4} \quad \text{and} \quad |A_1| \leq \frac{1}{6}.$$

Equalities hold for the bounds $|A_2|$, $|A_3|$ which is given by (3.10), (3.11) and using (5.2), (5.3).

Theorem 5.2 If $f \in \mathcal{SK}_{exp}$ is of the form (1.1), then

$$|A_3 - \eta A_2^2| \leq \max\left\{\frac{1}{6}, \left|\frac{3\eta - 2}{48}\right|\right\}.$$

This inequality is sharp.

Proof. From (5.5) and (5.6), we have

$$|A_3 - \eta A_2^2| = \frac{1}{6}\left|\xi_2 - \frac{1}{4}\xi_1^2 + \frac{3\eta}{8}\xi_1^2\right|$$

$$= \frac{1}{6}\left|\xi_2 + \left(\frac{3\eta - 2}{8}\right)\xi_1^2\right|.$$

Implementation of **Lemma 2.2** and triangle inequality, we obtain.

$$|A_3 - \eta A_2^2| \leq \max\left\{\frac{1}{6}, \left|\frac{3\eta - 2}{48}\right|\right\}.$$

Putting $\eta = 1$, we get the following corollary.

Corollary 5.3 Let $f \in \mathcal{SK}_{exp}$. Then

$$|A_3 - \eta A_2^2| \leq \frac{1}{6}.$$

Equality is determined by using (5.2), (5.3) and (3.11).

Theorem 5.4 If $f \in \mathcal{SK}_{exp}$ is of the form (1.1), then

$$|A_4 - A_2A_3| \leq \frac{1}{16}.$$

Proof. From (5.5), (5.6) and (5.7), we get

$$|A_4 - A_2A_3| = \frac{1}{16}\left|\xi_3 - \frac{7}{6}\xi_1\xi_2 - \frac{1}{6}\xi_1^3\right|,$$

and so taking $\sigma = -\frac{7}{6}$ and $\varsigma = -\frac{1}{6}$ in Lemma 2.1 yields

$$|A_4 - A_2A_3| \leq \frac{1}{16}.$$

Which is the required proof.

Theorem 5.5 Let $f \in \mathcal{SK}_{exp}$. Then

$$\left|\mathcal{H}_{2,2}(f^{-1})\right| = |A_2A_4 - A_3^2| \leq \frac{1}{36}.$$

This result is sharp. Equality is held by using (5.2), (5.3), (5.4) and (3.11).

Proof. From (5.5), (5.6) and (5.7), we have

$$|A_2A_4 - A_3^2| = \frac{1}{36}\left|\frac{1}{16}\xi_1^4 + \frac{17}{32}\xi_1^2\xi_2 - \frac{9}{16}\xi_1\xi_3 + \xi_2^2\right|$$
$$= \frac{1}{36}\left|\frac{1}{2}(\xi_2^2 - \xi_1\xi_3) + \frac{1}{2}\left(\frac{1}{8}\xi_1^4 - \frac{1}{8}\xi_1\xi_3 + \frac{17}{16}\xi_1^2\xi_2 + \xi_2^2\right)\right|$$
$$\leq \frac{1}{72}|\xi_2^2 - \xi_1\xi_3| + \frac{1}{72}\left|\frac{1}{8}\xi_1^4 - \frac{1}{8}\xi_1\xi_3 + \frac{17}{16}\xi_1^2\xi_2 + \xi_2^2\right|$$
$$= \frac{1}{72}R_1 + \frac{1}{72}R_2,$$

where

$$R_1 = |\xi_2^2 - \xi_1\xi_3|$$

and

$$R_2 = \left|\frac{1}{8}\xi_1^4 - \frac{1}{8}\xi_1\xi_3 + \frac{17}{16}\xi_1^2\xi_2 + \xi_2^2\right|$$

Using **Lemma 2.4**, we obtain $R_1 \leq 1$. For R_2 using **Lemma 2.3**, we have

$$|R_2| \leq \frac{1}{8}|\xi_1|^4 + \frac{1}{8}|\xi_1|\left(1 - |\xi_1|^2 - \frac{|\xi_2|^2}{1+|\xi_1|}\right) + \frac{17}{16}|\xi_1|^2|\xi_2| + |\xi_2|^2,$$
$$\leq \frac{1}{8}|\xi_1|^4 + \frac{1}{8}|\xi_1| - \frac{1}{8}|\xi_1|^3 - \frac{|\xi_1||\xi_2|^2}{8(1+|\xi_1|)} + \frac{17}{16}|\xi_1|^2|\xi_2| + |\xi_2|^2,$$
$$\leq \frac{1}{8}|\xi_1|^4 + \frac{1}{8}|\xi_1| - \frac{1}{8}|\xi_1|^3 + |\xi_2|^2\left(1 - \frac{|\xi_1|}{8(1+|\xi_1|)}\right) + \frac{17}{16}|\xi_1|^2|\xi_2|. \quad (5.8)$$

Since $\left(1 - \frac{|\xi_1|}{8(1+|\xi_1|)}\right) > 0$. Therefore, we can put $|\xi_2| \leq 1 - |\xi_1|^2$ in (5.8), we have

$$|R_2| \leq \frac{1}{8}|\xi_1|^4 + \frac{1}{8}|\xi_1| - \frac{1}{8}|\xi_1|^3 + (1-|\xi_1|^2)^2\left(1 - \frac{|\xi_1|}{8(1+|\xi_1|)}\right) + \frac{17}{16}|\xi_1|^2(1-|\xi_1|^2).$$

After elementary calculus of maxima and minima, we get

$$|R_2| \leq 1.$$

Hence

$$\left|\mathcal{H}_{2,2}(f^{-1})\right| \leq \frac{1}{72}R_1 + \frac{1}{72}R_2 \leq \frac{1}{36}.$$

Which is the required proof.

6 | CONCLUSION

We have studied a novel class of convex functions with symmetric points which are defined by subordinating to exponential function. Certain coefficient problems like the first four coefficient bounds, the second Hankel determinant, and Fekete-Szegö inequality for the said convex functions as well as for the coefficients of inverse and logarithmic functions are investigated. It is worth to mention that all the obtained results are sharp.

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