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Hamacher Prioritized Aggregation Operators Based on Complex Picture Fuzzy Sets and Their Applications in Decision-Making Problems

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ABSTRACT

Multi-attribute decision-making (MADM) technique is a dominant procedure for evaluating the best alternative between the collection of finite alternatives. Hamacher prioritized aggregation operators are very useful for aggregating the collection of information into a singleton set. Additionally, a complex picture fuzzy (CPF) set is very capable and reliable for depicting awkward and unreliable information in genuine life problems. In this manuscript, we present the CPF Hamacher prioritized averaging (CIFHPA) operator, CPF Hamacher prioritized ordered averaging (CIFHPOA) operator, CPF Hamacher prioritized geometric (CIFHPG) operator, and CPF Hamacher prioritized ordered geometric (CIFHPOG) operator. Furthermore, some remarkable properties of these operators are also examined. Moreover, we contribute to the improvement of the MADM procedure with a new plan of an algorithm in the CPF environment. Furthermore, we select and solve a MADM problem that evaluates the best optimal between the collection of decisions based on derived operators. Finally, we check the reliability and effectiveness of these operators with the help of a comparative analysis between proposed techniques and some prevailing operators.

Keywords: Complex picture fuzzy sets, Hamacher prioritized aggregation operators, Decision-making problems.

1 | INTRODUCTION

Decision-making procedure is involved in many fields, for instance, computer sciences, engineering sciences, software engineering, mathematical science, and many other different areas in the presence of classical set theory. During the decision-making procedure, we lose a lot of information because of incomplete information and in the presence of classical set theory, we have only two possibilities such as zero or one. For coping with such types of problems, Zadeh [1] introduced the fuzzy set (FS) theory. FS has a grade of membership "$\mathcal{Z}^{at}_{\Sigma_{CF}}(\mathfrak{g})$", where $\mathcal{Z}^{at}_{\Sigma_{CF}}(\mathfrak{g}) \in [0,1]$. Furthermore, FS copes with a membership grade but does not deal with the non-membership grade, where the non-membership grade is also their advantage. Therefore, Atanasov [2], [3] derived the intuitionistic FS (IFS). IFS has a grade of membership "$\mathcal{Z}^{at}_{\Sigma_{CF}}(\mathfrak{g})$", and a grade of non-membership "$\mathcal{Y}^{at}_{\Sigma_{CF}}(\mathfrak{g})$", where $\mathcal{Z}^{at}_{\Sigma_{CF}}(\mathfrak{g}) + \mathcal{Y}^{at}_{\Sigma_{CF}}(\mathfrak{g}) \in [0,1]$. IFS and FS both are valuable and reliable because of their structure, many people have utilized them in the environment of decision-making, medical diagnosis, clustering analysis, and machine learning. During the election, we faced multiple types of opinions such as yes, no, abstinence, and refusal, because some people have cast their vote in the favor (against, abstinence, and refusal) of someone. Where such types of problems cannot be handled from traditional IFS and FS. For this, Cuong [4], [5] invented the idea of picture FS (PFS). PFS has many grades, such as membership grade "$\mathcal{Z}_{\Sigma_{CF}}^{at}(\mathfrak{g})$", abstinence grade "$\mathfrak{D}_{\Sigma_{CF}}^{at}(\mathfrak{g})$", and non-membership grade "$\mathcal{Y}_{\Sigma_{CF}}^{at}(\mathfrak{g})$", with a prominent characteristic: $\mathcal{Z}_{\Sigma_{CF}}^{at}(\mathfrak{g}) + \mathfrak{D}_{\Sigma_{CF}}^{at}(\mathfrak{g}) + \mathcal{Y}_{\Sigma_{CF}}^{at}(\mathfrak{g}) \in [0,1]$. Many applications have been done by different scholars in the presence of PFSs, IFSs, and FSs.

Dealing with two-dimension information is a very challenging task because FS theory has to cope with one-dimension information. Therefore, computing the membership grade in the shape of a complex number is a very awkward and complicated task. Therefore, Ramot et al. [6] exposed the complex FS (CFS), where CFS copes with two-dimension information in the shape of a singleton set. CFS covered the membership in the form of a complex-valued membership grade, whose amplitude term and phase term are contained in the unit interval. Furthermore, Alkouri and Salleh [7] derived the complex IFS (CIFS). CIFS copes with membership grade "$\left(\mathcal{Z}_{\Sigma_{CF}}^{at}(\mathfrak{g}), \mathcal{Z}_{\Sigma_{CF}}^{pt}(\mathfrak{g})\right)$" and non-membership grade "$\left(\mathcal{Y}_{\Sigma_{CF}}^{at}(\mathfrak{g}), \mathcal{Y}_{\Sigma_{CF}}^{pt}(\mathfrak{g})\right)$" with a valuable condition: $\mathcal{Z}_{\Sigma_{CF}}^{at}(\mathfrak{g}) + \mathcal{Y}_{\Sigma_{CF}}^{at}(\mathfrak{g}) \in [0,1]$ and $\mathcal{Z}_{\Sigma_{CF}}^{pt}(\mathfrak{g}) + \mathcal{Y}_{\Sigma_{CF}}^{pt}(\mathfrak{g}) \in [0,1]$. Furthermore, FSs, IFSs, and CFSs are the special cases of the CIFSs. Many applications have been done by different scholars in the presence of CIFSs. Furthermore, utilizing the abstinence grade is very awkward and complicated, therefore, Akram et al. [8] examined the complex PFS (CPFS) which copes with membership "$\left(\mathcal{Z}_{\Sigma_{CF}}^{at}(\mathfrak{g}), \mathcal{Z}_{\Sigma_{CF}}^{pt}(\mathfrak{g})\right)$", abstinence "$\left(\mathfrak{D}_{\Sigma_{CF}}^{at}(\mathfrak{g}), \mathfrak{D}_{\Sigma_{CF}}^{pt}(\mathfrak{g})\right)$", and falsity "$\left(\mathcal{Y}_{\Sigma_{CF}}^{at}(\mathfrak{g}), \mathcal{Y}_{\Sigma_{CF}}^{pt}(\mathfrak{g})\right)$" grades with two same conditions: $\mathcal{Z}_{\Sigma_{CF}}^{at}(\mathfrak{g}) + \mathfrak{D}_{\Sigma_{CF}}^{at}(\mathfrak{g}) + \mathcal{Y}_{\Sigma_{CF}}^{at}(\mathfrak{g}) \in [0,1]$ and $\mathcal{Z}_{\Sigma_{CF}}^{pt}(\mathfrak{g}) + \mathfrak{D}_{\Sigma_{CF}}^{pt}(\mathfrak{g}) + \mathcal{Y}_{\Sigma_{CF}}^{pt}(\mathfrak{g}) \in [0,1]$. CPFSs are very valuable and reliable because of their structure, many people have utilized it in the environment of decision-making, medical diagnosis, clustering analysis, and machine learning.

Hamacher [9] invented the Hamacher t-norm and t-conorm in 1975, which is the most perfect and valuable extension of algebraic norms and is used for deriving the construction of any kind of aggregation operator. Furthermore, Yager [10] exposed the prioritized AOs (PAOs) for aggregating the collection of information. Moreover, Yager [11] derived the AOs for fuzzy information in 1994. Additionally, Xu [12] presented the AOs for IFSs. Xu and Yager [13] evaluated the geometric AOs for IFSs. Wang et al. [14] presented the geometric AOs for PFSs. Garg [15] examined the AOs for PFSs. Jana and Pal [16] evaluated the Hamacher operators for PFSs. Huang [17] presented the Hamacher operators for IFSs. Wei [18] exposed the Hamacher operators for PFSs. Yu [19] invented the PAOs for generalized IFSs. The major contribution of this paper is listed below:

1. To present the CIFHPA operator, CIFHPOA operator, CIFHPG operator, and CIFHPOG operator. Furthermore, some remarkable properties of these operators are also examined.

2. To contribute to the improvement of the MADM procedure with a new plan of an algorithm in the CPF environment.
3. To select and solve a MADM problem that evaluates the best optimal between the collection of decisions based on derived operators.
4. To check the reliability and effectiveness of these operators with the help of a comparative analysis between proposed techniques and some prevailing operators.

This manuscript is arranged below: In Section 2, we evaluated the CPFSs, algebraic operational laws, Hamacher operational laws, and PAOs. In Section 3, we presented the CIFHPA operator, CIFHPOA operator, CIFHPG operator, and CIFHPOG operator. Furthermore, some remarkable properties of these operators are also examined. In Section 4, we contributed to the improvement of the MADM procedure with a new plan of an algorithm in the CPF environment. Furthermore, we select and solve a MADM problem that evaluates the best optimal between the collection of decisions based on derived operators. In Section 5, we checked the reliability and effectiveness of these operators with the help of a comparative analysis between proposed techniques and some prevailing operators. Some concluding remarks are stated in Section 6.

2 | PRELIMINARIES

In this section, we evaluate the CPFSs, algebraic operational laws, Hamacher operational laws, and prioritized averaging (PA) operators.

Definition 1: [8] For a universe π_u, CPFS Σ_{CF} is an expression of the shape:

$$\Sigma_{CF} = \left\{\left(\mathcal{Z}_{\Sigma_{CF}}(\mathrm{g}), \mathfrak{O}_{\Sigma_{CF}}(\mathrm{g}), \mathcal{Y}_{\Sigma_{CF}}(\mathrm{g})\right) : \mathrm{g} \in \pi_u\right\}$$

Furthermore, we explained the truth, abstinence, and falsity grades, such as $\mathcal{Z}_{\Sigma_{CF}}(\mathrm{g}) = \left(\mathcal{Z}^{at}_{\Sigma_{CF}}(\mathrm{g}), \mathcal{Z}^{pt}_{\Sigma_{CF}}(\mathrm{g})\right), \mathfrak{O}_{\Sigma_{CF}}(\mathrm{g}) = \left(\mathfrak{O}^{at}_{\Sigma_{CF}}(\mathrm{g}), \mathfrak{O}^{pt}_{\Sigma_{CF}}(\mathrm{g})\right)$ and $\mathcal{Y}_{\Sigma_{CF}}(\mathrm{g}) = \left(\mathcal{Y}^{at}_{\Sigma_{CF}}(\mathrm{g}), \mathcal{Y}^{pt}_{\Sigma_{CF}}(\mathrm{g})\right)$ with a valuable condition, that is $0 \leq \mathcal{Z}^{at}_{\Sigma_{CF}}(\mathrm{g}) + \mathfrak{O}^{at}_{\Sigma_{CF}}(\mathrm{g}) + \mathcal{Y}^{at}_{\Sigma_{CF}}(\mathrm{g}) \leq 1$ and $0 \leq \mathcal{Z}^{pt}_{\Sigma_{CF}}(\mathrm{g}) + \mathfrak{O}^{pt}_{\Sigma_{CF}}(\mathrm{g}) + \mathcal{Y}^{pt}_{\Sigma_{CF}}(\mathrm{g}) \leq 1$. Moreover, we stated the refusal grade, such as $r_{\Sigma_{CF}}(\mathrm{g}) = \left(r^{at}_{\Sigma_{CF}}(\mathrm{g}), r^{pt}_{\Sigma_{CF}}(\mathrm{g})\right) = \left(1 - \left(\mathcal{Z}^{at}_{\Sigma_{CF}}(\mathrm{g}) + \mathfrak{O}^{at}_{\Sigma_{CF}}(\mathrm{g}) + \mathcal{Y}^{at}_{\Sigma_{CF}}(\mathrm{g})\right), 1 - \left(\mathcal{Z}^{pt}_{\Sigma_{CF}}(\mathrm{g}) + \mathfrak{O}^{pt}_{\Sigma_{CF}}(\mathrm{g}) + \mathcal{Y}^{pt}_{\Sigma_{CF}}(\mathrm{g})\right)\right)$. Finally, we described the simple form of CPFN, such as: $\Sigma^j_{CF} = \left(\left(\mathcal{Z}^{at}_{\Sigma_j}, \mathcal{Z}^{pt}_{\Sigma_j}\right), \left(\mathfrak{O}^{at}_{\Sigma_j}, \mathfrak{O}^{pt}_{\Sigma_j}\right), \left(\mathcal{Y}^{at}_{\Sigma_j}, \mathcal{Y}^{pt}_{\Sigma_j}\right)\right), j = 1,2, \dots, w$.

Definition 2: [8] Consider $\Sigma^j_{CF} = \left(\left(\mathcal{Z}^{at}_{\Sigma_j}, \mathcal{Z}^{pt}_{\Sigma_j}\right), \left(\mathfrak{O}^{at}_{\Sigma_j}, \mathfrak{O}^{pt}_{\Sigma_j}\right), \left(\mathcal{Y}^{at}_{\Sigma_j}, \mathcal{Y}^{pt}_{\Sigma_j}\right)\right), j = 1,2$, thus

$$\Sigma^1_{CF} \oplus \Sigma^2_{CF} = \left(\left(\mathcal{Z}^{at}_{\Sigma_1} + \mathcal{Z}^{at}_{\Sigma_2} - \mathcal{Z}^{at}_{\Sigma_1}\mathcal{Z}^{at}_{\Sigma_2}, \mathcal{Z}^{pt}_{\Sigma_1} + \mathcal{Z}^{pt}_{\Sigma_2} - \mathcal{Z}^{pt}_{\Sigma_1}\mathcal{Z}^{pt}_{\Sigma_2}\right), \left(\mathfrak{O}^{at}_{\Sigma_1}\mathfrak{O}^{at}_{\Sigma_2}, \mathfrak{O}^{pt}_{\Sigma_1}\mathfrak{O}^{pt}_{\Sigma_2}\right), \left(\mathcal{Y}^{at}_{\Sigma_1}\mathcal{Y}^{at}_{\Sigma_2}, \mathcal{Y}^{pt}_{\Sigma_1}\mathcal{Y}^{pt}_{\Sigma_2}\right)\right)$$

$$\Sigma_{CF}^{1} \otimes \Sigma_{CF}^{2} = \begin{pmatrix} \left(\mathcal{Z}_{\Sigma_1}^{at}\mathcal{Z}_{\Sigma_2}^{at}, \mathcal{Z}_{\Sigma_1}^{pt}\mathcal{Z}_{\Sigma_2}^{pt}\right), \left(\mathfrak{D}_{\Sigma_1}^{at} + \mathfrak{D}_{\Sigma_2}^{at} - \mathfrak{D}_{\Sigma_1}^{at}\mathfrak{D}_{\Sigma_2}^{at}, \mathfrak{D}_{\Sigma_1}^{pt} + \mathfrak{D}_{\Sigma_2}^{pt} - \mathfrak{D}_{\Sigma_1}^{pt}\mathfrak{D}_{\Sigma_2}^{pt}\right), \\ \left(\mathcal{Y}_{\Sigma_1}^{at} + \mathcal{Y}_{\Sigma_2}^{at} - \mathcal{Y}_{\Sigma_1}^{at}\mathcal{Y}_{\Sigma_2}^{at}, \mathcal{Y}_{\Sigma_1}^{pt} + \mathcal{Y}_{\Sigma_2}^{pt} - \mathcal{Y}_{\Sigma_1}^{pt}\mathcal{Y}_{\Sigma_2}^{pt}\right) \end{pmatrix}$$

$$\phi^{s}\Sigma_{CF}^{1} = \left(\left(1 - \left(1 - \mathcal{Z}_{\Sigma_1}^{at}\right)^{\phi^{s}}, 1 - \left(1 - \mathcal{Z}_{\Sigma_1}^{pt}\right)^{\phi^{s}}\right), \left(\left(\mathfrak{D}_{\Sigma_1}^{at}\right)^{\phi^{s}}, \left(\mathfrak{D}_{\Sigma_1}^{pt}\right)^{\phi^{s}}\right), \left(\left(\mathcal{Y}_{\Sigma_1}^{at}\right)^{\phi^{s}}, \left(\mathcal{Y}_{\Sigma_1}^{pt}\right)^{\phi^{s}}\right)\right)$$

$$\left(\Sigma_{CF}^{1}\right)^{\phi^{s}} = \begin{pmatrix} \left(\left(\mathcal{Z}_{\Sigma_1}^{at}\right)^{\phi^{s}}, \left(\mathcal{Z}_{\Sigma_1}^{pt}\right)^{\phi^{s}}\right), \left(1 - \left(1 - \mathfrak{D}_{\Sigma_1}^{at}\right)^{\phi^{s}}, 1 - \left(1 - \mathfrak{D}_{\Sigma_1}^{pt}\right)^{\phi^{s}}\right), \\ \left(1 - \left(1 - \mathcal{Y}_{\Sigma_1}^{at}\right)^{\phi^{s}}, 1 - \left(1 - \mathcal{Y}_{\Sigma_1}^{pt}\right)^{\phi^{s}}\right) \end{pmatrix}$$

Definition 3: [8] Consider $\Sigma_{CF}^{j} = \left(\left(\mathcal{Z}_{\Sigma_j}^{at}, \mathcal{Z}_{\Sigma_j}^{pt}\right), \left(\mathfrak{D}_{\Sigma_j}^{at}, \mathfrak{D}_{\Sigma_j}^{pt}\right), \left(\mathcal{Y}_{\Sigma_j}^{at}, \mathcal{Y}_{\Sigma_j}^{pt}\right)\right), j = 1,2$, thus

$$S\left(\Sigma_{CF}^{1}\right) = \frac{1}{2}\left(\left(\mathcal{Z}_{\Sigma_j}^{at} + \mathcal{Z}_{\Sigma_j}^{pt}\right) - \left(\mathfrak{D}_{\Sigma_j}^{at} + \mathfrak{D}_{\Sigma_j}^{pt}\right) - \left(\mathcal{Y}_{\Sigma_j}^{at} + \mathcal{Y}_{\Sigma_j}^{pt}\right)\right) \in [-1,1]$$

$$H\left(\Sigma_{CF}^{1}\right) = \frac{1}{2}\left(\left(\mathcal{Z}_{\Sigma_j}^{at} + \mathcal{Z}_{\Sigma_j}^{pt}\right) + \left(\mathfrak{D}_{\Sigma_j}^{at} + \mathfrak{D}_{\Sigma_j}^{pt}\right) + \left(\mathcal{Y}_{\Sigma_j}^{at} + \mathcal{Y}_{\Sigma_j}^{pt}\right)\right) \in [0,1]$$

Where, the information in $S(\Sigma_{CF}^{1})$ and $H(\Sigma_{CF}^{1})$ states the score and accuracy values with the following rules, such as, if $S(\Sigma_{CF}^{1}) > S(\Sigma_{CF}^{2}) \Rightarrow \Sigma_{CF}^{1} > \Sigma_{CF}^{2}$, if $S(\Sigma_{CF}^{1}) < S(\Sigma_{CF}^{2}) \Rightarrow \Sigma_{CF}^{1} < \Sigma_{CF}^{2}$, if $S(\Sigma_{CF}^{1}) = S(\Sigma_{CF}^{2}) \Rightarrow$, thus if $H(\Sigma_{CF}^{1}) > H(\Sigma_{CF}^{2}) \Rightarrow \Sigma_{CF}^{1} > \Sigma_{CF}^{2}$, if $H(\Sigma_{CF}^{1}) < H(\Sigma_{CF}^{2}) \Rightarrow \Sigma_{CF}^{1} < \Sigma_{CF}^{2}$.

Definition 4: [8] Consider $\Sigma_{CF}^{j} = \left(\left(\mathcal{Z}_{\Sigma_j}^{at}, \mathcal{Z}_{\Sigma_j}^{pt}\right), \left(\mathfrak{D}_{\Sigma_j}^{at}, \mathfrak{D}_{\Sigma_j}^{pt}\right), \left(\mathcal{Y}_{\Sigma_j}^{at}, \mathcal{Y}_{\Sigma_j}^{pt}\right)\right), j = 1,2$, thus

$$\Sigma_{CF}^{1} \oplus \Sigma_{CF}^{2} = \begin{pmatrix} \left(\frac{\mathcal{Z}_{\Sigma_1}^{at} + \mathcal{Z}_{\Sigma_2}^{at} - \mathcal{Z}_{\Sigma_1}^{at}\mathcal{Z}_{\Sigma_2}^{at} - (1-\phi^{s})\mathcal{Z}_{\Sigma_1}^{at}\mathcal{Z}_{\Sigma_2}^{at}}{1 - (1-\phi^{s})\mathcal{Z}_{\Sigma_1}^{at}\mathcal{Z}_{\Sigma_2}^{at}}, \frac{\mathcal{Z}_{\Sigma_1}^{pt} + \mathcal{Z}_{\Sigma_2}^{pt} - \mathcal{Z}_{\Sigma_1}^{pt}\mathcal{Z}_{\Sigma_2}^{pt} - (1-\phi^{s})\mathcal{Z}_{\Sigma_1}^{pt}\mathcal{Z}_{\Sigma_2}^{pt}}{1 - (1-\phi^{s})\mathcal{Z}_{\Sigma_1}^{pt}\mathcal{Z}_{\Sigma_2}^{pt}}\right), \\ \left(\frac{\mathfrak{D}_{\Sigma_1}^{at}\mathfrak{D}_{\Sigma_2}^{at}}{\phi^{s} + (1-\phi^{s})\left(\mathfrak{D}_{\Sigma_1}^{at} + \mathfrak{D}_{\Sigma_2}^{at} - \mathfrak{D}_{\Sigma_1}^{at}\mathfrak{D}_{\Sigma_2}^{at}\right)}, \frac{\mathfrak{D}_{\Sigma_1}^{pt}\mathfrak{D}_{\Sigma_2}^{pt}}{\phi^{s} + (1-\phi^{s})\left(\mathfrak{D}_{\Sigma_1}^{pt} + \mathfrak{D}_{\Sigma_2}^{pt} - \mathfrak{D}_{\Sigma_1}^{pt}\mathfrak{D}_{\Sigma_2}^{pt}\right)}\right), \\ \left(\frac{\mathcal{Y}_{\Sigma_1}^{at}\mathcal{Y}_{\Sigma_2}^{at}}{\phi^{s} + (1-\phi^{s})\left(\mathcal{Y}_{\Sigma_1}^{at} + \mathcal{Y}_{\Sigma_2}^{at} - \mathcal{Y}_{\Sigma_1}^{at}\mathcal{Y}_{\Sigma_2}^{at}\right)}, \frac{\mathcal{Y}_{\Sigma_1}^{pt}\mathcal{Y}_{\Sigma_2}^{pt}}{\phi^{s} + (1-\phi^{s})\left(\mathcal{Y}_{\Sigma_1}^{pt} + \mathcal{Y}_{\Sigma_2}^{pt} - \mathcal{Y}_{\Sigma_1}^{pt}\mathcal{Y}_{\Sigma_2}^{pt}\right)}\right) \end{pmatrix}$$

$$\Sigma_{CF}^{1} \oplus \Sigma_{CF}^{2} = \begin{pmatrix} \left(\frac{\mathcal{Z}_{\Sigma_1}^{at}\mathcal{Z}_{\Sigma_2}^{at}}{\phi^{s} + (1-\phi^{s})\left(\mathcal{Z}_{\Sigma_1}^{at} + \mathcal{Z}_{\Sigma_2}^{at} - \mathcal{Z}_{\Sigma_1}^{at}\mathcal{Z}_{\Sigma_2}^{at}\right)}, \frac{\mathcal{Z}_{\Sigma_1}^{pt}\mathcal{Z}_{\Sigma_2}^{pt}}{\phi^{s} + (1-\phi^{s})\left(\mathcal{Z}_{\Sigma_1}^{pt} + \mathcal{Z}_{\Sigma_2}^{pt} - \mathcal{Z}_{\Sigma_1}^{pt}\mathcal{Z}_{\Sigma_2}^{pt}\right)}\right), \\ \left(\frac{\mathfrak{D}_{\Sigma_1}^{at} + \mathfrak{D}_{\Sigma_2}^{at} - \mathfrak{D}_{\Sigma_1}^{at}\mathfrak{D}_{\Sigma_2}^{at} - (1-\phi^{s})\mathfrak{D}_{\Sigma_1}^{at}\mathfrak{D}_{\Sigma_2}^{at}}{1 - (1-\phi^{s})\mathfrak{D}_{\Sigma_1}^{at}\mathfrak{D}_{\Sigma_2}^{at}}, \frac{\mathfrak{D}_{\Sigma_1}^{pt} + \mathfrak{D}_{\Sigma_2}^{pt} - \mathfrak{D}_{\Sigma_1}^{pt}\mathfrak{D}_{\Sigma_2}^{pt} - (1-\phi^{s})\mathfrak{D}_{\Sigma_1}^{pt}\mathfrak{D}_{\Sigma_2}^{pt}}{1 - (1-\phi^{s})\mathfrak{D}_{\Sigma_1}^{pt}\mathfrak{D}_{\Sigma_2}^{pt}}\right), \\ \left(\frac{\mathcal{Y}_{\Sigma_1}^{at} + \mathcal{Y}_{\Sigma_2}^{at} - \mathcal{Y}_{\Sigma_1}^{at}\mathcal{Y}_{\Sigma_2}^{at} - (1-\phi^{s})\mathcal{Y}_{\Sigma_1}^{at}\mathcal{Y}_{\Sigma_2}^{at}}{1 - (1-\phi^{s})\mathcal{Y}_{\Sigma_1}^{at}\mathcal{Y}_{\Sigma_2}^{at}}, \frac{\mathcal{Y}_{\Sigma_1}^{pt} + \mathcal{Y}_{\Sigma_2}^{pt} - \mathcal{Y}_{\Sigma_1}^{pt}\mathcal{Y}_{\Sigma_2}^{pt} - (1-\phi^{s})\mathcal{Y}_{\Sigma_1}^{pt}\mathcal{Y}_{\Sigma_2}^{pt}}{1 - (1-\phi^{s})\mathcal{Y}_{\Sigma_1}^{pt}\mathcal{Y}_{\Sigma_2}^{pt}}\right) \end{pmatrix}$$

- $$\nabla^s\Sigma_{CF}^1=\left(\begin{array}{c}\left(\frac{\left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_1}^{at}\right)^{\nabla^s}-\left(1-\mathcal{Z}_{\Sigma_1}^{at}\right)^{\nabla^s}}{\left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_1}^{at}\right)^{\nabla^s}+(\phi^s-1)\left(1-\mathcal{Z}_{\Sigma_1}^{at}\right)^{\nabla^s}},\frac{\left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_1}^{pt}\right)^{\nabla^s}-\left(1-\mathcal{Z}_{\Sigma_1}^{pt}\right)^{\nabla^s}}{\left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_1}^{pt}\right)^{\nabla^s}+(\phi^s-1)\left(1-\mathcal{Z}_{\Sigma_1}^{pt}\right)^{\nabla^s}}\right),\\ \left(\frac{(\phi^s)\left(\mathfrak{D}_{\Sigma_1}^{at}\right)^{\nabla^s}}{\left(1+(\phi^s-1)\left(1-\mathfrak{D}_{\Sigma_1}^{at}\right)\right)^{\nabla^s}+(\phi^s-1)\left(\mathfrak{D}_{\Sigma_1}^{at}\right)^{\nabla^s}},\frac{(\phi^s)\left(\mathfrak{D}_{\Sigma_1}^{pt}\right)^{\nabla^s}}{\left(1+(\phi^s-1)\left(1-\mathfrak{D}_{\Sigma_1}^{pt}\right)\right)^{\nabla^s}+(\phi^s-1)\left(\mathfrak{D}_{\Sigma_1}^{pt}\right)^{\nabla^s}}\right),\\ \left(\frac{(\phi^s)\left(\mathcal{Y}_{\Sigma_1}^{at}\right)^{\nabla^s}}{\left(1+(\phi^s-1)\left(1-\mathcal{Y}_{\Sigma_1}^{at}\right)\right)^{\nabla^s}+(\phi^s-1)\left(\mathcal{Y}_{\Sigma_1}^{at}\right)^{\nabla^s}},\frac{(\phi^s)\left(\mathcal{Y}_{\Sigma_1}^{pt}\right)^{\nabla^s}}{\left(1+(\phi^s-1)\left(1-\mathcal{Y}_{\Sigma_1}^{pt}\right)\right)^{\nabla^s}+(\phi^s-1)\left(\mathcal{Y}_{\Sigma_1}^{pt}\right)^{\nabla^s}}\right)\end{array}\right)$$

$$\left(\Sigma_{CF}^1\right)^{\nabla^s}$$
$$=\left(\begin{array}{c}\left(\frac{(\phi^s)\left(\mathcal{Z}_{\Sigma_1}^{at}\right)^{\nabla^s}}{\left(1+(\phi^s-1)\left(1-\mathcal{Z}_{\Sigma_1}^{at}\right)\right)^{\nabla^s}+(\phi^s-1)\left(\mathcal{Z}_{\Sigma_1}^{at}\right)^{\nabla^s}},\frac{(\phi^s)\left(\mathcal{Z}_{\Sigma_1}^{pt}\right)^{\nabla^s}}{\left(1+(\phi^s-1)\left(1-\mathcal{Z}_{\Sigma_1}^{pt}\right)\right)^{\nabla^s}+(\phi^s-1)\left(\mathcal{Z}_{\Sigma_1}^{pt}\right)^{\nabla^s}}\right),\\ \left(\frac{\left(1+(\phi^s-1)\mathfrak{D}_{\Sigma_1}^{at}\right)^{\nabla^s}-\left(1-\mathfrak{D}_{\Sigma_1}^{at}\right)^{\nabla^s}}{\left(1+(\phi^s-1)\mathfrak{D}_{\Sigma_1}^{at}\right)^{\nabla^s}+(\phi^s-1)\left(1-\mathfrak{D}_{\Sigma_1}^{at}\right)^{\nabla^s}},\frac{\left(1+(\phi^s-1)\mathfrak{D}_{\Sigma_1}^{pt}\right)^{\nabla^s}-\left(1-\mathfrak{D}_{\Sigma_1}^{pt}\right)^{\nabla^s}}{\left(1+(\phi^s-1)\mathfrak{D}_{\Sigma_1}^{pt}\right)^{\nabla^s}+(\phi^s-1)\left(1-\mathfrak{D}_{\Sigma_1}^{pt}\right)^{\nabla^s}}\right),\\ \left(\frac{\left(1+(\phi^s-1)\mathcal{Y}_{\Sigma_1}^{at}\right)^{\nabla^s}-\left(1-\mathcal{Y}_{\Sigma_1}^{at}\right)^{\nabla^s}}{\left(1+(\phi^s-1)\mathcal{Y}_{\Sigma_1}^{at}\right)^{\nabla^s}+(\phi^s-1)\left(1-\mathcal{Y}_{\Sigma_1}^{at}\right)^{\nabla^s}},\frac{\left(1+(\phi^s-1)\mathcal{Y}_{\Sigma_1}^{pt}\right)^{\nabla^s}-\left(1-\mathcal{Y}_{\Sigma_1}^{pt}\right)^{\nabla^s}}{\left(1+(\phi^s-1)\mathcal{Y}_{\Sigma_1}^{pt}\right)^{\nabla^s}+(\phi^s-1)\left(1-\mathcal{Y}_{\Sigma_1}^{pt}\right)^{\nabla^s}}\right)\end{array}\right)$$

Definition 5: [10] Consider that $\Sigma_{CF}^j, j=1,2,\dots,w$ be the collection of attributes with a linear ordering $\Sigma_{CF}^1 > \Sigma_{CF}^2 > \cdots > \Sigma_{CF}^w$, where $\Sigma_{CF}^j \in [0,1]$, then the prioritized averaging operator is stated below:

$$PA\left(\Sigma_{CF}^j\right)=\sum_{j=1}^{w}\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}\Sigma_{CF}^j$$

Where, $\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}$ represented the priority degree with $\overline{\overline{\Xi}}_J=\prod_{r=1}^{j-1}S\left(\Sigma_{CF}^r\right)$ and $\overline{\overline{\Xi}_1}=1$.

3 | HAMACHER PRIORITIZED AGGREGATION OPERATORS BASED ON CPFSS

In this section, we derive the CPFHPA operator, CPFHPOA operator, CPFHPG operator, and CPFHPOG operator, and their remarkable properties.

Definition 6: Consider $\Sigma_{CF}^j=\left(\left(\mathcal{Z}_{\Sigma_j}^{at},\mathcal{Z}_{\Sigma_j}^{pt}\right),\left(\mathfrak{D}_{\Sigma_j}^{at},\mathfrak{D}_{\Sigma_j}^{pt}\right),\left(\mathcal{Y}_{\Sigma_j}^{at},\mathcal{Y}_{\Sigma_j}^{pt}\right)\right), j=1,2,\dots,w$, thus the CPFHPA operator is exposed in the shape:

$$CPFHPA\left(\Sigma_{CF}^1,\Sigma_{CF}^2,\dots,\Sigma_{CF}^w\right)=\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}\Sigma_{CF}^1\oplus\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}\Sigma_{CF}^2\oplus\dots\oplus\frac{\overline{\overline{\Xi_w}}}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}\Sigma_{CF}^w=\oplus_{j=1}^{w}\left(\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}\Sigma_{CF}^j\right)$$

Where, $\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}$ represented the priority degree with $\overline{\overline{\Xi}}_J=\prod_{r=1}^{j-1}S\left(\Sigma_{CF}^r\right)$ and $\overline{\overline{\Xi_1}}=1$.

Theorem 1: Consider $\Sigma_{CF}^j=\left(\left(\mathcal{Z}_{\Sigma_j}^{at},\mathcal{Z}_{\Sigma_j}^{pt}\right),\left(\mathfrak{D}_{\Sigma_j}^{at},\mathfrak{D}_{\Sigma_j}^{pt}\right),\left(\mathcal{Y}_{\Sigma_j}^{at},\mathcal{Y}_{\Sigma_j}^{pt}\right)\right), j=1,2,\dots,w$, thus by using the information in Definition 6, we evaluate that the aggregated value of Definition 6 is again a CIFN, such as

$$CPFHPA(\Sigma_{CF}^{1},\Sigma_{CF}^{2},\dots,\Sigma_{CF}^{w})=\left(\begin{array}{c}\left(\begin{array}{c}\frac{\prod_{j=1}^{w}\left(1+(\phi^{s}-1)\mathcal{Z}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}-\prod_{j=1}^{w}\left(1-\mathcal{Z}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w}\left(1+(\phi^{s}-1)\mathcal{Z}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\prod_{j=1}^{w}\left(1-\mathcal{Z}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}},\\ \frac{\prod_{j=1}^{w}\left(1+(\phi^{s}-1)\mathcal{Z}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}-\prod_{j=1}^{w}\left(1-\mathcal{Z}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w}\left(1+(\phi^{s}-1)\mathcal{Z}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\prod_{j=1}^{w}\left(1-\mathcal{Z}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}\end{array}\right),\\ \left(\begin{array}{c}\frac{(\phi^{s})\prod_{j=1}^{w}\left(\mathfrak{D}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w}\left(1+(\phi^{s}-1)\left(1-\mathfrak{D}_{\Sigma_j}^{at}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\prod_{j=1}^{w}\left(\mathfrak{D}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}},\\ \frac{(\phi^{s})\prod_{j=1}^{w}\left(\mathfrak{D}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w}\left(1+(\phi^{s}-1)\left(1-\mathfrak{D}_{\Sigma_j}^{pt}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\prod_{j=1}^{w}\left(\mathfrak{D}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}\end{array}\right),\\ \left(\begin{array}{c}\frac{(\phi^{s})\prod_{j=1}^{w}\left(\mathcal{Y}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w}\left(1+(\phi^{s}-1)\left(1-\mathcal{Y}_{\Sigma_j}^{at}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\prod_{j=1}^{w}\left(\mathcal{Y}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}},\\ \frac{(\phi^{s})\prod_{j=1}^{w}\left(\mathcal{Y}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w}\left(1+(\phi^{s}-1)\left(1-\mathcal{Y}_{\Sigma_j}^{pt}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\prod_{j=1}^{w}\left(\mathcal{Y}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}\end{array}\right)\end{array}\right)$$

Proof: With the help of Mathematical induction, we derive the above information, for this, if $w=2$, then

$$\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}\Sigma_{CF}^{1}=\left(\begin{array}{c}\left(\begin{array}{c}\dfrac{\left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_1}^{at}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}-\left(1-\mathcal{Z}_{\Sigma_1}^{at}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_1}^{at}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^s-1)\left(1-\mathcal{Z}_{\Sigma_1}^{at}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}},\\ \dfrac{\left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_1}^{pt}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}-\left(1-\mathcal{Z}_{\Sigma_1}^{pt}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_1}^{pt}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^s-1)\left(1-\mathcal{Z}_{\Sigma_1}^{pt}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}\end{array}\right),\\ \left(\begin{array}{c}\dfrac{(\phi^s)\left(\mathfrak{D}_{\Sigma_1}^{at}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^s-1)\left(1-\mathfrak{D}_{\Sigma_1}^{at}\right)\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^s-1)\left(\mathfrak{D}_{\Sigma_1}^{at}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}},\\ \dfrac{(\phi^s)\left(\mathfrak{D}_{\Sigma_1}^{pt}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^s-1)\left(1-\mathfrak{D}_{\Sigma_1}^{pt}\right)\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^s-1)\left(\mathfrak{D}_{\Sigma_1}^{pt}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}\end{array}\right),\\ \left(\begin{array}{c}\dfrac{(\phi^s)\left(\mathcal{Y}_{\Sigma_1}^{at}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^s-1)\left(1-\mathcal{Y}_{\Sigma_1}^{at}\right)\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^s-1)\left(\mathcal{Y}_{\Sigma_1}^{at}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}},\\ \dfrac{(\phi^s)\left(\mathcal{Y}_{\Sigma_1}^{pt}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^s-1)\left(1-\mathcal{Y}_{\Sigma_1}^{pt}\right)\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^s-1)\left(\mathcal{Y}_{\Sigma_1}^{pt}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}\end{array}\right)\end{array}\right)$$

$$\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}\Sigma_{CF}^{2}=\left(\begin{array}{c}\left(\begin{array}{c}\dfrac{\left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_2}^{at}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}-\left(1-\mathcal{Z}_{\Sigma_2}^{at}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_2}^{at}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^s-1)\left(1-\mathcal{Z}_{\Sigma_2}^{at}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}},\\ \dfrac{\left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_2}^{pt}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}-\left(1-\mathcal{Z}_{\Sigma_2}^{pt}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_2}^{pt}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^s-1)\left(1-\mathcal{Z}_{\Sigma_2}^{pt}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}\end{array}\right),\\ \left(\begin{array}{c}\dfrac{(\phi^s)\left(\mathfrak{D}_{\Sigma_2}^{at}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^s-1)\left(1-\mathfrak{D}_{\Sigma_2}^{at}\right)\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^s-1)\left(\mathfrak{D}_{\Sigma_2}^{at}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}},\\ \dfrac{(\phi^s)\left(\mathfrak{D}_{\Sigma_2}^{pt}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^s-1)\left(1-\mathfrak{D}_{\Sigma_2}^{pt}\right)\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^s-1)\left(\mathfrak{D}_{\Sigma_2}^{pt}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}\end{array}\right),\\ \left(\begin{array}{c}\dfrac{(\phi^s)\left(\mathcal{Y}_{\Sigma_2}^{at}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^s-1)\left(1-\mathcal{Y}_{\Sigma_2}^{at}\right)\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^s-1)\left(\mathcal{Y}_{\Sigma_2}^{at}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}},\\ \dfrac{(\phi^s)\left(\mathcal{Y}_{\Sigma_2}^{pt}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^s-1)\left(1-\mathcal{Y}_{\Sigma_2}^{pt}\right)\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^s-1)\left(\mathcal{Y}_{\Sigma_2}^{pt}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}\end{array}\right)\end{array}\right)$$

Thus,

$$CPFHPA(\Sigma_{CF}^{1}, \Sigma_{CF}^{2}) = \frac{\overline{\overline{\overline{\Xi}}}_1}{\sum_{j=1}^{w} \overline{\overline{\overline{\Xi}}}_J} \Sigma_{CF}^{1} \oplus \frac{\overline{\overline{\overline{\Xi}}}_2}{\sum_{j=1}^{w} \overline{\overline{\overline{\Xi}}}_J} \Sigma_{CF}^{2}$$

$$
=\left(\begin{array}{c}
\left(\begin{array}{c}
\dfrac{\left(1+(\phi^{s}-1)\mathcal{Z}_{\Sigma_1}^{at}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}-\left(1-\mathcal{Z}_{\Sigma_1}^{at}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^{s}-1)\mathcal{Z}_{\Sigma_1}^{at}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\left(1-\mathcal{Z}_{\Sigma_1}^{at}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}},\\
\dfrac{\left(1+(\phi^{s}-1)\mathcal{Z}_{\Sigma_1}^{pt}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}-\left(1-\mathcal{Z}_{\Sigma_1}^{pt}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^{s}-1)\mathcal{Z}_{\Sigma_1}^{pt}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\left(1-\mathcal{Z}_{\Sigma_1}^{pt}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}
\end{array}\right),\\
\left(\begin{array}{c}
\dfrac{(\phi^{s})\left(\mathfrak{D}_{\Sigma_1}^{at}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^{s}-1)\left(1-\mathfrak{D}_{\Sigma_1}^{at}\right)\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\left(\mathfrak{D}_{\Sigma_1}^{at}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}},\\
\dfrac{(\phi^{s})\left(\mathfrak{D}_{\Sigma_1}^{pt}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^{s}-1)\left(1-\mathfrak{D}_{\Sigma_1}^{pt}\right)\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\left(\mathfrak{D}_{\Sigma_1}^{pt}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}
\end{array}\right),\\
\left(\begin{array}{c}
\dfrac{(\phi^{s})\left(\mathcal{Y}_{\Sigma_1}^{at}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^{s}-1)\left(1-\mathcal{Y}_{\Sigma_1}^{at}\right)\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\left(\mathcal{Y}_{\Sigma_1}^{at}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}},\\
\dfrac{(\phi^{s})\left(\mathcal{Y}_{\Sigma_1}^{pt}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^{s}-1)\left(1-\mathcal{Y}_{\Sigma_1}^{pt}\right)\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\left(\mathcal{Y}_{\Sigma_1}^{pt}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}
\end{array}\right)
\end{array}\right)
$$

$$
\oplus\left(\begin{array}{c}
\left(\begin{array}{c}
\dfrac{\left(1+(\phi^{s}-1)\mathcal{Z}_{\Sigma_2}^{at}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}-\left(1-\mathcal{Z}_{\Sigma_2}^{at}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^{s}-1)\mathcal{Z}_{\Sigma_2}^{at}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\left(1-\mathcal{Z}_{\Sigma_2}^{at}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}},\\
\dfrac{\left(1+(\phi^{s}-1)\mathcal{Z}_{\Sigma_2}^{pt}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}-\left(1-\mathcal{Z}_{\Sigma_2}^{pt}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^{s}-1)\mathcal{Z}_{\Sigma_2}^{pt}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\left(1-\mathcal{Z}_{\Sigma_2}^{pt}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}
\end{array}\right),\\
\left(\begin{array}{c}
\dfrac{(\phi^{s})\left(\mathfrak{D}_{\Sigma_2}^{at}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^{s}-1)\left(1-\mathfrak{D}_{\Sigma_2}^{at}\right)\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\left(\mathfrak{D}_{\Sigma_2}^{at}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}},\\
\dfrac{(\phi^{s})\left(\mathfrak{D}_{\Sigma_2}^{pt}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^{s}-1)\left(1-\mathfrak{D}_{\Sigma_2}^{pt}\right)\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\left(\mathfrak{D}_{\Sigma_2}^{pt}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}
\end{array}\right),\\
\left(\begin{array}{c}
\dfrac{(\phi^{s})\left(\mathcal{Y}_{\Sigma_2}^{at}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^{s}-1)\left(1-\mathcal{Y}_{\Sigma_2}^{at}\right)\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\left(\mathcal{Y}_{\Sigma_2}^{at}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}},\\
\dfrac{(\phi^{s})\left(\mathcal{Y}_{\Sigma_2}^{pt}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}{\left(1+(\phi^{s}-1)\left(1-\mathcal{Y}_{\Sigma_2}^{pt}\right)\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\left(\mathcal{Y}_{\Sigma_2}^{pt}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{2}\overline{\overline{\Xi_J}}}}}
\end{array}\right)
\end{array}\right)
$$

$$
= \left(\begin{array}{c}
\left(\begin{array}{c}
\dfrac{\prod_{j=1}^{2}\left(1+(\phi^{s}-1)\mathcal{Z}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}-\prod_{j=1}^{2}\left(1-\mathcal{Z}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{2}\left(1+(\phi^{s}-1)\mathcal{Z}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\prod_{j=1}^{2}\left(1-\mathcal{Z}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}, \\
\dfrac{\prod_{j=1}^{2}\left(1+(\phi^{s}-1)\mathcal{Z}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}-\prod_{j=1}^{2}\left(1-\mathcal{Z}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{2}\left(1+(\phi^{s}-1)\mathcal{Z}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\prod_{j=1}^{2}\left(1-\mathcal{Z}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}
\end{array} \right), \\
\left(\begin{array}{c}
\dfrac{(\phi^{s})\prod_{j=1}^{2}\left(\mathfrak{D}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{2}\left(1+(\phi^{s}-1)\left(1-\mathfrak{D}_{\Sigma_j}^{at}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\prod_{j=1}^{2}\left(\mathfrak{D}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}, \\
\dfrac{(\phi^{s})\prod_{j=1}^{2}\left(\mathfrak{D}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{2}\left(1+(\phi^{s}-1)\left(1-\mathfrak{D}_{\Sigma_j}^{pt}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\prod_{j=1}^{2}\left(\mathfrak{D}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}
\end{array} \right), \\
\left(\begin{array}{c}
\dfrac{(\phi^{s})\prod_{j=1}^{2}\left(\mathcal{Y}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{2}\left(1+(\phi^{s}-1)\left(1-\mathcal{Y}_{\Sigma_j}^{at}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\prod_{j=1}^{2}\left(\mathcal{Y}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}, \\
\dfrac{(\phi^{s})\prod_{j=1}^{2}\left(\mathcal{Y}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{2}\left(1+(\phi^{s}-1)\left(1-\mathcal{Y}_{\Sigma_j}^{pt}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\prod_{j=1}^{2}\left(\mathcal{Y}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}
\end{array} \right)
\end{array} \right)
$$

hence, for $w = 2$, we successfully get our target, furthermore, we consider that the proposed theory holds for $w = w'$, thus.

$$CPFHPA\left(\Sigma_{CF}^{1},\Sigma_{CF}^{2},\dots,\Sigma_{CF}^{w'}\right)=\left(\begin{array}{c}\left(\begin{array}{c}\dfrac{\prod_{j=1}^{w'}\left(1+(\phi^{s}-1)\mathcal{Z}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}-\prod_{j=1}^{w'}\left(1-\mathcal{Z}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w'}\left(1+(\phi^{s}-1)\mathcal{Z}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\prod_{j=1}^{w'}\left(1-\mathcal{Z}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}},\\ \dfrac{\prod_{j=1}^{w'}\left(1+(\phi^{s}-1)\mathcal{Z}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}-\prod_{j=1}^{w'}\left(1-\mathcal{Z}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w'}\left(1+(\phi^{s}-1)\mathcal{Z}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\prod_{j=1}^{w'}\left(1-\mathcal{Z}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}\end{array}\right),\\ \left(\begin{array}{c}\dfrac{(\phi^{s})\prod_{j=1}^{w'}\left(\mathfrak{D}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w'}\left(1+(\phi^{s}-1)\left(1-\mathfrak{D}_{\Sigma_j}^{at}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\prod_{j=1}^{w'}\left(\mathfrak{D}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}},\\ \dfrac{(\phi^{s})\prod_{j=1}^{w'}\left(\mathfrak{D}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w'}\left(1+(\phi^{s}-1)\left(1-\mathfrak{D}_{\Sigma_j}^{pt}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\prod_{j=1}^{w'}\left(\mathfrak{D}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}\end{array}\right),\\ \left(\begin{array}{c}\dfrac{(\phi^{s})\prod_{j=1}^{w'}\left(\mathcal{Y}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w'}\left(1+(\phi^{s}-1)\left(1-\mathcal{Y}_{\Sigma_j}^{at}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\prod_{j=1}^{w'}\left(\mathcal{Y}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}},\\ \dfrac{(\phi^{s})\prod_{j=1}^{w'}\left(\mathcal{Y}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w'}\left(1+(\phi^{s}-1)\left(1-\mathcal{Y}_{\Sigma_j}^{pt}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^{s}-1)\prod_{j=1}^{w'}\left(\mathcal{Y}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}\end{array}\right)\end{array}\right)$$

Then, we prove that the proposed theory also holds for $w=w'+1$, such as

$$\begin{aligned}CPFHPA\left(\Sigma_{CF}^{1},\Sigma_{CF}^{2},\dots,\Sigma_{CF}^{w}\right)&=\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}\Sigma_{CF}^{1}\oplus\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}\Sigma_{CF}^{2}\oplus\dots\oplus\frac{\overline{\overline{\Xi_{w'}}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}\Sigma_{CF}^{w'}\oplus\frac{\overline{\overline{\Xi_{w'+1}}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}\Sigma_{CF}^{w'+1}\\ &=\oplus_{j=1}^{w'}\left(\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}\Sigma_{CF}^{j}\right)\oplus\frac{\overline{\overline{\Xi_{w'+1}}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}\Sigma_{CF}^{w'+1}\end{aligned}$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \dfrac{\prod_{j=1}^{w'} \left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}} - \prod_{j=1}^{w'} \left(1-\mathcal{Z}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w'} \left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}} + (\phi^s-1)\prod_{j=1}^{w'} \left(1-\mathcal{Z}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}, \\ \dfrac{\prod_{j=1}^{w'} \left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}} - \prod_{j=1}^{w'} \left(1-\mathcal{Z}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w'} \left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}} + (\phi^s-1)\prod_{j=1}^{w'} \left(1-\mathcal{Z}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}} \end{array} \right), \\ \left(\begin{array}{c} \dfrac{(\phi^s)\prod_{j=1}^{w'} \left(\mathfrak{D}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w'} \left(1+(\phi^s-1)\left(1-\mathfrak{D}_{\Sigma_j}^{at}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}} + (\phi^s-1)\prod_{j=1}^{w'} \left(\mathfrak{D}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}, \\ \dfrac{(\phi^s)\prod_{j=1}^{w'} \left(\mathfrak{D}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w'} \left(1+(\phi^s-1)\left(1-\mathfrak{D}_{\Sigma_j}^{pt}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}} + (\phi^s-1)\prod_{j=1}^{w'} \left(\mathfrak{D}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}} \end{array} \right), \\ \left(\begin{array}{c} \dfrac{(\phi^s)\prod_{j=1}^{w'} \left(\mathcal{Y}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w'} \left(1+(\phi^s-1)\left(1-\mathcal{Y}_{\Sigma_j}^{at}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}} + (\phi^s-1)\prod_{j=1}^{w'} \left(\mathcal{Y}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}, \\ \dfrac{(\phi^s)\prod_{j=1}^{w'} \left(\mathcal{Y}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w'} \left(1+(\phi^s-1)\left(1-\mathcal{Y}_{\Sigma_j}^{pt}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}} + (\phi^s-1)\prod_{j=1}^{w'} \left(\mathcal{Y}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}} \end{array} \right) \end{array} \right) \oplus \frac{\overline{\overline{\Xi_{w'+1}}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}\Sigma_{CF}^{w'+1}$$

$$= \left(\begin{array}{c} \left(\dfrac{\prod_{j=1}^{w'+1}\left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}} - \prod_{j=1}^{w'+1}\left(1-\mathcal{Z}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w'+1}\left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}} + (\phi^s-1)\prod_{j=1}^{w'+1}\left(1-\mathcal{Z}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}}}, \dfrac{\prod_{j=1}^{w'+1}\left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}} - \prod_{j=1}^{w'+1}\left(1-\mathcal{Z}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w'+1}\left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}} + (\phi^s-1)\prod_{j=1}^{w'+1}\left(1-\mathcal{Z}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}}} \right), \\ \left(\dfrac{(\phi^s)\prod_{j=1}^{w'+1}\left(\mathfrak{D}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w'+1}\left(1+(\phi^s-1)\left(1-\mathfrak{D}_{\Sigma_j}^{at}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}} + (\phi^s-1)\prod_{j=1}^{w'+1}\left(\mathfrak{D}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}}}, \dfrac{(\phi^s)\prod_{j=1}^{w'+1}\left(\mathfrak{D}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w'+1}\left(1+(\phi^s-1)\left(1-\mathfrak{D}_{\Sigma_j}^{pt}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}} + (\phi^s-1)\prod_{j=1}^{w'+1}\left(\mathfrak{D}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}}} \right), \\ \left(\dfrac{(\phi^s)\prod_{j=1}^{w'+1}\left(\mathcal{Y}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w'+1}\left(1+(\phi^s-1)\left(1-\mathcal{Y}_{\Sigma_j}^{at}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}} + (\phi^s-1)\prod_{j=1}^{w'+1}\left(\mathcal{Y}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}}}, \dfrac{(\phi^s)\prod_{j=1}^{w'+1}\left(\mathcal{Y}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w'+1}\left(1+(\phi^s-1)\left(1-\mathcal{Y}_{\Sigma_j}^{pt}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}} + (\phi^s-1)\prod_{j=1}^{w'+1}\left(\mathcal{Y}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w'+1}\overline{\overline{\Xi_J}}}}} \right) \end{array} \right)$$

Hence, the invented operator is held for all positive values of w.

Property 1: Consider $\Sigma_{CF}^{j} = \left(\left(\mathcal{Z}_{\Sigma_j}^{at}, \mathcal{Z}_{\Sigma_j}^{pt}\right), \left(\mathfrak{D}_{\Sigma_j}^{at}, \mathfrak{D}_{\Sigma_j}^{pt}\right), \left(\mathcal{Y}_{\Sigma_j}^{at}, \mathcal{Y}_{\Sigma_j}^{pt}\right)\right), j = 1,2, \dots, w$ be the collection of CPFNs, thus

1) (Idempotency) If $\Sigma_{CF}^{j} = \Sigma_{CF} = \left(\left(\mathcal{Z}_{\Sigma}^{at}, \mathcal{Z}_{\Sigma}^{pt}\right), \left(\mathfrak{D}_{\Sigma}^{at}, \mathfrak{D}_{\Sigma}^{pt}\right), \left(\mathcal{Y}_{\Sigma}^{at}, \mathcal{Y}_{\Sigma}^{pt}\right)\right), j = 1,2, \dots, w$, thus
$$CPFHPA(\Sigma_{CF}^{1}, \Sigma_{CF}^{2}, \dots, \Sigma_{CF}^{w}) = \Sigma_{CF}.$$

2) (Monotonicity) If $\Sigma_{CF}^{j} \leq {\Sigma^{\#}}_{CF}^{j}$, thus
$$CPFHPA(\Sigma_{CF}^{1}, \Sigma_{CF}^{2}, \dots, \Sigma_{CF}^{w}) \leq CPFHPA\left({\Sigma^{\#}}_{CF}^{1}, {\Sigma^{\#}}_{CF}^{2}, \dots, {\Sigma^{\#}}_{CF}^{w}\right).$$

3) (Boundedness) If $\Sigma_{CF}^{-} = \left(\left(\min_j \mathcal{Z}_{\Sigma_j}^{at}, \min_j \mathcal{Z}_{\Sigma_j}^{pt}\right), \left(\min_j \mathfrak{D}_{\Sigma_j}^{at}, \min_j \mathfrak{D}_{\Sigma_j}^{pt}\right), \left(\max_j \mathcal{Y}_{\Sigma_j}^{at}, \max_j \mathcal{Y}_{\Sigma_j}^{pt}\right)\right)$ and $\Sigma_{CF}^{+} = \left(\left(\max_j \mathcal{Z}_{\Sigma_j}^{at}, \max_j \mathcal{Z}_{\Sigma_j}^{pt}\right), \left(\min_j \mathfrak{D}_{\Sigma_j}^{at}, \min_j \mathfrak{D}_{\Sigma_j}^{pt}\right), \left(\min_j \mathcal{Y}_{\Sigma_j}^{at}, \min_j \mathcal{Y}_{\Sigma_j}^{pt}\right)\right)$, thus
$$\Sigma_{CF}^{-} \leq CPFHPA(\Sigma_{CF}^{1}, \Sigma_{CF}^{2}, \dots, \Sigma_{CF}^{w}) \leq \Sigma_{CF}^{+}.$$

Definition 7: Consider $\Sigma_{CF}^{j}=\left(\left(\mathcal{Z}_{\Sigma_j}^{at},\mathcal{Z}_{\Sigma_j}^{pt}\right),\left(\mathfrak{O}_{\Sigma_j}^{at},\mathfrak{O}_{\Sigma_j}^{pt}\right),\left(\mathcal{Y}_{\Sigma_j}^{at},\mathcal{Y}_{\Sigma_j}^{pt}\right)\right), j=1,2,\dots,w,$ thus the CPFHPOA operator is exposed in the shape:

$$CPFHPOA(\Sigma_{CF}^{1},\Sigma_{CF}^{2},\dots,\Sigma_{CF}^{w})=\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}\Sigma_{CF}^{o(1)}\oplus\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}\Sigma_{CF}^{o(2)}\oplus\dots\oplus\frac{\overline{\overline{\Xi_w}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}\Sigma_{CF}^{o(w)}=\oplus_{j=1}^{w}\left(\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}\Sigma_{CF}^{o(j)}\right)$$

Where, $\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}$ represented the priority degree with $\overline{\overline{\Xi_J}}=\prod_{r=1}^{j-1}S(\Sigma_{CF}^{r})$ and $\overline{\overline{\Xi_1}}=1$ with order $o(j)\le o(j-1)$.

Theorem 2: Consider $\Sigma_{CF}^{j}=\left(\left(\mathcal{Z}_{\Sigma_j}^{at},\mathcal{Z}_{\Sigma_j}^{pt}\right),\left(\mathfrak{O}_{\Sigma_j}^{at},\mathfrak{O}_{\Sigma_j}^{pt}\right),\left(\mathcal{Y}_{\Sigma_j}^{at},\mathcal{Y}_{\Sigma_j}^{pt}\right)\right), j=1,2,\dots,w,$ thus by using the information in Definition 7, we evaluate that the aggregated value of Definition 7 is again a CIFN, such as

$$CPFHPOA(\Sigma_{CF}^{1},\Sigma_{CF}^{2},\dots,\Sigma_{CF}^{w})$$
$$=\left(\begin{array}{c}\left(\begin{array}{c}\frac{\prod_{j=1}^{w}\left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_{o(j)}}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}-\prod_{j=1}^{w}\left(1-\mathcal{Z}_{\Sigma_{o(j)}}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w}\left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_{o(j)}}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^s-1)\prod_{j=1}^{w}\left(1-\mathcal{Z}_{\Sigma_{o(j)}}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}},\\ \frac{\prod_{j=1}^{w}\left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_{o(j)}}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}-\prod_{j=1}^{w}\left(1-\mathcal{Z}_{\Sigma_{o(j)}}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w}\left(1+(\phi^s-1)\mathcal{Z}_{\Sigma_{o(j)}}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^s-1)\prod_{j=1}^{w}\left(1-\mathcal{Z}_{\Sigma_{o(j)}}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}\end{array}\right),\\ \left(\begin{array}{c}\frac{(\phi^s)\prod_{j=1}^{w}\left(\mathfrak{O}_{\Sigma_{o(j)}}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w}\left(1+(\phi^s-1)\left(1-\mathfrak{O}_{\Sigma_{o(j)}}^{at}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^s-1)\prod_{j=1}^{w}\left(\mathfrak{O}_{\Sigma_{o(j)}}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}},\\ \frac{(\phi^s)\prod_{j=1}^{w}\left(\mathfrak{O}_{\Sigma_{o(j)}}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w}\left(1+(\phi^s-1)\left(1-\mathfrak{O}_{\Sigma_{o(j)}}^{pt}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^s-1)\prod_{j=1}^{w}\left(\mathfrak{O}_{\Sigma_{o(j)}}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}\end{array}\right),\\ \left(\begin{array}{c}\frac{(\phi^s)\prod_{j=1}^{w}\left(\mathcal{Y}_{\Sigma_{o(j)}}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w}\left(1+(\phi^s-1)\left(1-\mathcal{Y}_{\Sigma_{o(j)}}^{at}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^s-1)\prod_{j=1}^{w}\left(\mathcal{Y}_{\Sigma_{o(j)}}^{at}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}},\\ \frac{(\phi^s)\prod_{j=1}^{w}\left(\mathcal{Y}_{\Sigma_{o(j)}}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}{\prod_{j=1}^{w}\left(1+(\phi^s-1)\left(1-\mathcal{Y}_{\Sigma_{o(j)}}^{pt}\right)\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}+(\phi^s-1)\prod_{j=1}^{w}\left(\mathcal{Y}_{\Sigma_{o(j)}}^{pt}\right)^{\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}}}\end{array}\right)\end{array}\right)$$

Property 2: Consider $\Sigma_{CF}^{j}=\left(\left(\mathcal{Z}_{\Sigma_j}^{at},\mathcal{Z}_{\Sigma_j}^{pt}\right),\left(\mathfrak{O}_{\Sigma_j}^{at},\mathfrak{O}_{\Sigma_j}^{pt}\right),\left(\mathcal{Y}_{\Sigma_j}^{at},\mathcal{Y}_{\Sigma_j}^{pt}\right)\right), j=1,2,\dots,w$ be the collection of CPFNs, thus

1) (Idempotency) If $\Sigma_{CF}^{j}=\Sigma_{CF}=\left(\left(\mathcal{Z}_{\Sigma}^{at},\mathcal{Z}_{\Sigma}^{pt}\right),\left(\mathfrak{O}_{\Sigma}^{at},\mathfrak{O}_{\Sigma}^{pt}\right),\left(\mathcal{Y}_{\Sigma}^{at},\mathcal{Y}_{\Sigma}^{pt}\right)\right), j=1,2,\dots,w,$ thus
$$CPFHPOA(\Sigma_{CF}^{1},\Sigma_{CF}^{2},\dots,\Sigma_{CF}^{w})=\Sigma_{CF}.$$

2) (Monotonicity) If $\Sigma_{CF}^{j} \leq \Sigma{\#}_{CF}^{j}$, thus

$$CPFHPOA(\Sigma_{CF}^{1}, \Sigma_{CF}^{2}, \dots, \Sigma_{CF}^{w}) \leq CPFHPOA\left(\Sigma{\#}_{CF}^{1}, \Sigma{\#}_{CF}^{2}, \dots, \Sigma{\#}_{CF}^{w}\right).$$

3) (Boundedness) If $\Sigma_{CF}^{-} = \left(\left(\min_j \mathcal{Z}_{\Sigma_j}^{at}, \min_j \mathcal{Z}_{\Sigma_j}^{pt}\right), \left(\min_j \mathfrak{O}_{\Sigma_j}^{at}, \min_j \mathfrak{O}_{\Sigma_j}^{pt}\right), \left(\max_j \mathcal{Y}_{\Sigma_j}^{at}, \max_j \mathcal{Y}_{\Sigma_j}^{pt}\right)\right)$ and $\Sigma_{CF}^{+} = \left(\left(\max_j \mathcal{Z}_{\Sigma_j}^{at}, \max_j \mathcal{Z}_{\Sigma_j}^{pt}\right), \left(\min_j \mathfrak{O}_{\Sigma_j}^{at}, \min_j \mathfrak{O}_{\Sigma_j}^{pt}\right), \left(\min_j \mathcal{Y}_{\Sigma_j}^{at}, \min_j \mathcal{Y}_{\Sigma_j}^{pt}\right)\right)$, thus

$$\Sigma_{CF}^{-} \leq CPFHPOA(\Sigma_{CF}^{1}, \Sigma_{CF}^{2}, \dots, \Sigma_{CF}^{w}) \leq \Sigma_{CF}^{+}.$$

Definition 8: Consider $\Sigma_{CF}^{j} = \left(\left(\mathcal{Z}_{\Sigma_j}^{at}, \mathcal{Z}_{\Sigma_j}^{pt}\right), \left(\mathfrak{O}_{\Sigma_j}^{at}, \mathfrak{O}_{\Sigma_j}^{pt}\right), \left(\mathcal{Y}_{\Sigma_j}^{at}, \mathcal{Y}_{\Sigma_j}^{pt}\right)\right), j = 1,2, \dots, w$, thus the CPFHPG operator is exposed in the shape:

$$CPFHPG(\Sigma_{CF}^{1}, \Sigma_{CF}^{2}, \dots, \Sigma_{CF}^{w}) = (\Sigma_{CF}^{1})^{\frac{\overline{\overline{\Xi}}_1}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}} \otimes (\Sigma_{CF}^{2})^{\frac{\overline{\overline{\Xi}}_2}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}} \otimes \dots \otimes (\Sigma_{CF}^{w})^{\frac{\overline{\overline{\Xi}}_w}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}} = \otimes_{j=1}^{w} (\Sigma_{CF}^{j})^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}$$

Where, $\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}$ represented the priority degree with $\overline{\overline{\Xi}}_J = \prod_{r=1}^{j-1} S(\Sigma_{CF}^{r})$ and $\overline{\overline{\Xi}}_1 = 1$.

Theorem 3: Consider $\Sigma_{CF}^{j} = \left(\left(\mathcal{Z}_{\Sigma_j}^{at}, \mathcal{Z}_{\Sigma_j}^{pt}\right), \left(\mathfrak{O}_{\Sigma_j}^{at}, \mathfrak{O}_{\Sigma_j}^{pt}\right), \left(\mathcal{Y}_{\Sigma_j}^{at}, \mathcal{Y}_{\Sigma_j}^{pt}\right)\right), j = 1,2, \dots, w,$ thus by using the information in Definition 8, we evaluate that the aggregated value of Definition 8 is again a CIFN, such as

$$CPFHPG(\Sigma_{CF}^{1}, \Sigma_{CF}^{2}, \dots, \Sigma_{CF}^{w}) = \left(\begin{array}{c}
\left(\begin{array}{c}
\frac{(\phi^s)\prod_{j=1}^{w}\left(\mathcal{Z}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}}{\prod_{j=1}^{w}\left(1+(\phi^s-1)\left(1-\mathcal{Z}_{\Sigma_j}^{at}\right)\right)^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}+(\phi^s-1)\prod_{j=1}^{w}\left(\mathcal{Z}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}}, \\
\frac{(\phi^s)\prod_{j=1}^{w}\left(\mathcal{Z}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}}{\prod_{j=1}^{w}\left(1+(\phi^s-1)\left(1-\mathcal{Z}_{\Sigma_j}^{pt}\right)\right)^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}+(\phi^s-1)\prod_{j=1}^{w}\left(\mathcal{Z}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}}
\end{array}\right), \\
\left(\begin{array}{c}
\frac{\prod_{j=1}^{w}\left(1+(\phi^s-1)\mathfrak{O}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}-\prod_{j=1}^{w}\left(1-\mathfrak{O}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}}{\prod_{j=1}^{w}\left(1+(\phi^s-1)\mathfrak{O}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}+(\phi^s-1)\prod_{j=1}^{w}\left(1-\mathfrak{O}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}}, \\
\frac{\prod_{j=1}^{w}\left(1+(\phi^s-1)\mathfrak{O}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}-\prod_{j=1}^{w}\left(1-\mathfrak{O}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}}{\prod_{j=1}^{w}\left(1+(\phi^s-1)\mathfrak{O}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}+(\phi^s-1)\prod_{j=1}^{w}\left(1-\mathfrak{O}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}}
\end{array}\right), \\
\left(\begin{array}{c}
\frac{\prod_{j=1}^{w}\left(1+(\phi^s-1)\mathcal{Y}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}-\prod_{j=1}^{w}\left(1-\mathcal{Y}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}}{\prod_{j=1}^{w}\left(1+(\phi^s-1)\mathcal{Y}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}+(\phi^s-1)\prod_{j=1}^{w}\left(1-\mathcal{Y}_{\Sigma_j}^{at}\right)^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}}, \\
\frac{\prod_{j=1}^{w}\left(1+(\phi^s-1)\mathcal{Y}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}-\prod_{j=1}^{w}\left(1-\mathcal{Y}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}}{\prod_{j=1}^{w}\left(1+(\phi^s-1)\mathcal{Y}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}+(\phi^s-1)\prod_{j=1}^{w}\left(1-\mathcal{Y}_{\Sigma_j}^{pt}\right)^{\frac{\overline{\overline{\Xi}}_J}{\sum_{j=1}^{w}\overline{\overline{\Xi}}_J}}}
\end{array}\right)
\end{array}\right)$$

Property 3: Consider $\Sigma_{CF}^{j}=\left(\left(\mathcal{Z}_{\Sigma_j}^{at},\mathcal{Z}_{\Sigma_j}^{pt}\right),\left(\mathfrak{D}_{\Sigma_j}^{at},\mathfrak{D}_{\Sigma_j}^{pt}\right),\left(\mathcal{Y}_{\Sigma_j}^{at},\mathcal{Y}_{\Sigma_j}^{pt}\right)\right), j=1,2,\dots,w$ be the collection of CPFNs, thus

1) (Idempotency) If $\Sigma_{CF}^{j}=\Sigma_{CF}=\left(\left(\mathcal{Z}_{\Sigma}^{at},\mathcal{Z}_{\Sigma}^{pt}\right),\left(\mathfrak{D}_{\Sigma}^{at},\mathfrak{D}_{\Sigma}^{pt}\right),\left(\mathcal{Y}_{\Sigma}^{at},\mathcal{Y}_{\Sigma}^{pt}\right)\right), j=1,2,\dots,w$, thus
$$CPFHPG(\Sigma_{CF}^{1},\Sigma_{CF}^{2},\dots,\Sigma_{CF}^{w})=\Sigma_{CF}.$$
2) (Monotonicity) If $\Sigma_{CF}^{j}\le\Sigma^{\#}{}_{CF}^{j}$, thus
$$CPFHPG(\Sigma_{CF}^{1},\Sigma_{CF}^{2},\dots,\Sigma_{CF}^{w})\le CPFHPG\left(\Sigma^{\#}{}_{CF}^{1},\Sigma^{\#}{}_{CF}^{2},\dots,\Sigma^{\#}{}_{CF}^{w}\right).$$
3) (Boundedness) If $\Sigma_{CF}^{-}=\left(\left(\min_j \mathcal{Z}_{\Sigma_j}^{at},\min_j \mathcal{Z}_{\Sigma_j}^{pt}\right),\left(\min_j \mathfrak{D}_{\Sigma_j}^{at},\min_j \mathfrak{D}_{\Sigma_j}^{pt}\right),\left(\max_j \mathcal{Y}_{\Sigma_j}^{at},\max_j \mathcal{Y}_{\Sigma_j}^{pt}\right)\right)$ and $\Sigma_{CF}^{+}=\left(\left(\max_j \mathcal{Z}_{\Sigma_j}^{at},\max_j \mathcal{Z}_{\Sigma_j}^{pt}\right),\left(\min_j \mathfrak{D}_{\Sigma_j}^{at},\min_j \mathfrak{D}_{\Sigma_j}^{pt}\right),\left(\min_j \mathcal{Y}_{\Sigma_j}^{at},\min_j \mathcal{Y}_{\Sigma_j}^{pt}\right)\right)$, thus
$$\Sigma_{CF}^{-}\le CPFHPG(\Sigma_{CF}^{1},\Sigma_{CF}^{2},\dots,\Sigma_{CF}^{w})\le\Sigma_{CF}^{+}.$$

Definition 9: Consider $\Sigma_{CF}^{j}=\left(\left(\mathcal{Z}_{\Sigma_j}^{at},\mathcal{Z}_{\Sigma_j}^{pt}\right),\left(\mathfrak{D}_{\Sigma_j}^{at},\mathfrak{D}_{\Sigma_j}^{pt}\right),\left(\mathcal{Y}_{\Sigma_j}^{at},\mathcal{Y}_{\Sigma_j}^{pt}\right)\right), j=1,2,\dots,w$, thus the CPFHPOG operator is exposed in the shape:

$$CPFHPOG(\Sigma_{CF}^{1},\Sigma_{CF}^{2},\dots,\Sigma_{CF}^{w})=\left(\Sigma_{CF}^{o(1)}\right)^{\frac{\overline{\overline{\Xi_1}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}\otimes\left(\Sigma_{CF}^{o(2)}\right)^{\frac{\overline{\overline{\Xi_2}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}\otimes\dots\otimes\left(\Sigma_{CF}^{o(w)}\right)^{\frac{\overline{\overline{\Xi_w}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}=\otimes_{j=1}^{w}\left(\Sigma_{CF}^{o(j)}\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}$$

Where, $\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}$ represented the priority degree with $\overline{\overline{\Xi_j}}=\prod_{r=1}^{j-1}S(\Sigma_{CF}^{r})$ and $\overline{\overline{\Xi_1}}=1$ with $o(j)\le o(j-1)$.

Theorem 4: Consider $\Sigma_{CF}^{j}=\left(\left(\mathcal{Z}_{\Sigma_j}^{at},\mathcal{Z}_{\Sigma_j}^{pt}\right),\left(\mathfrak{D}_{\Sigma_j}^{at},\mathfrak{D}_{\Sigma_j}^{pt}\right),\left(\mathcal{Y}_{\Sigma_j}^{at},\mathcal{Y}_{\Sigma_j}^{pt}\right)\right), j=1,2,\dots,w$, thus by using the information in Definition 9, we evaluate that the aggregated value of Definition 9 is again a CIFN, such as

$$CPFHPOG(\Sigma_{CF}^{1},\Sigma_{CF}^{2},\dots,\Sigma_{CF}^{w})$$
$$=\left(\begin{array}{c}\left(\begin{array}{c}\frac{(\phi^s)\prod_{j=1}^{w}\left(\mathcal{Z}_{\Sigma_{o(j)}}^{at}\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}}{\prod_{j=1}^{w}\left(1+(\phi^s-1)\left(1-\mathcal{Z}_{\Sigma_{o(j)}}^{at}\right)\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}+(\phi^s-1)\prod_{j=1}^{w}\left(\mathcal{Z}_{\Sigma_{o(j)}}^{at}\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}},\\ \frac{(\phi^s)\prod_{j=1}^{w}\left(\mathcal{Z}_{\Sigma_{o(j)}}^{pt}\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}}{\prod_{j=1}^{w}\left(1+(\phi^s-1)\left(1-\mathcal{Z}_{\Sigma_{o(j)}}^{pt}\right)\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}+(\phi^s-1)\prod_{j=1}^{w}\left(\mathcal{Z}_{\Sigma_{o(j)}}^{pt}\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}}\end{array}\right),\\ \left(\begin{array}{c}\frac{\prod_{j=1}^{w}\left(1+(\phi^s-1)\mathfrak{D}_{\Sigma_{o(j)}}^{at}\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}-\prod_{j=1}^{w}\left(1-\mathfrak{D}_{\Sigma_{o(j)}}^{at}\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}}{\prod_{j=1}^{w}\left(1+(\phi^s-1)\mathfrak{D}_{\Sigma_{o(j)}}^{at}\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}+(\phi^s-1)\prod_{j=1}^{w}\left(1-\mathfrak{D}_{\Sigma_{o(j)}}^{at}\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}},\\ \frac{\prod_{j=1}^{w}\left(1+(\phi^s-1)\mathfrak{D}_{\Sigma_{o(j)}}^{pt}\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}-\prod_{j=1}^{w}\left(1-\mathfrak{D}_{\Sigma_{o(j)}}^{pt}\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}}{\prod_{j=1}^{w}\left(1+(\phi^s-1)\mathfrak{D}_{\Sigma_{o(j)}}^{pt}\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}+(\phi^s-1)\prod_{j=1}^{w}\left(1-\mathfrak{D}_{\Sigma_{o(j)}}^{pt}\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}}\end{array}\right),\\ \left(\begin{array}{c}\frac{\prod_{j=1}^{w}\left(1+(\phi^s-1)\mathcal{Y}_{\Sigma_{o(j)}}^{at}\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}-\prod_{j=1}^{w}\left(1-\mathcal{Y}_{\Sigma_{o(j)}}^{at}\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}}{\prod_{j=1}^{w}\left(1+(\phi^s-1)\mathcal{Y}_{\Sigma_{o(j)}}^{at}\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}+(\phi^s-1)\prod_{j=1}^{w}\left(1-\mathcal{Y}_{\Sigma_{o(j)}}^{at}\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}},\\ \frac{\prod_{j=1}^{w}\left(1+(\phi^s-1)\mathcal{Y}_{\Sigma_{o(j)}}^{pt}\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}-\prod_{j=1}^{w}\left(1-\mathcal{Y}_{\Sigma_{o(j)}}^{pt}\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}}{\prod_{j=1}^{w}\left(1+(\phi^s-1)\mathcal{Y}_{\Sigma_{o(j)}}^{pt}\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}+(\phi^s-1)\prod_{j=1}^{w}\left(1-\mathcal{Y}_{\Sigma_{o(j)}}^{pt}\right)^{\frac{\overline{\overline{\Xi_j}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_j}}}}}\end{array}\right)\end{array}\right)$$

Property 4: Consider $\Sigma_{CF}^{j}=\left(\left(\mathcal{Z}_{\Sigma_j}^{at},\mathcal{Z}_{\Sigma_j}^{pt}\right),\left(\mathfrak{O}_{\Sigma_j}^{at},\mathfrak{O}_{\Sigma_j}^{pt}\right),\left(\mathcal{Y}_{\Sigma_j}^{at},\mathcal{Y}_{\Sigma_j}^{pt}\right)\right),j=1,2,\dots,w$ be the collection of CPFNs, thus

1) (Idempotency) If $\Sigma_{CF}^{j}=\Sigma_{CF}=\left(\left(\mathcal{Z}_{\Sigma}^{at},\mathcal{Z}_{\Sigma}^{pt}\right),\left(\mathfrak{O}_{\Sigma}^{at},\mathfrak{O}_{\Sigma}^{pt}\right),\left(\mathcal{Y}_{\Sigma}^{at},\mathcal{Y}_{\Sigma}^{pt}\right)\right),j=1,2,\dots,w$, thus
$$CPFHPOG(\Sigma_{CF}^{1},\Sigma_{CF}^{2},\dots,\Sigma_{CF}^{w})=\Sigma_{CF}.$$
2) (Monotonicity) If $\Sigma_{CF}^{j}\leq\Sigma^{\#}{}_{CF}^{j}$, thus
$$CPFHPOG(\Sigma_{CF}^{1},\Sigma_{CF}^{2},\dots,\Sigma_{CF}^{w})\leq CPFHPOG\left(\Sigma^{\#}{}_{CF}^{1},\Sigma^{\#}{}_{CF}^{2},\dots,\Sigma^{\#}{}_{CF}^{w}\right).$$
3) (Boundedness) If $\Sigma_{CF}^{-}=\left(\left(\min_{j}\mathcal{Z}_{\Sigma_j}^{at},\min_{j}\mathcal{Z}_{\Sigma_j}^{pt}\right),\left(\min_{j}\mathfrak{O}_{\Sigma_j}^{at},\min_{j}\mathfrak{O}_{\Sigma_j}^{pt}\right),\left(\max_{j}\mathcal{Y}_{\Sigma_j}^{at},\max_{j}\mathcal{Y}_{\Sigma_j}^{pt}\right)\right)$ and $\Sigma_{CF}^{+}=\left(\left(\max_{j}\mathcal{Z}_{\Sigma_j}^{at},\max_{j}\mathcal{Z}_{\Sigma_j}^{pt}\right),\left(\min_{j}\mathfrak{O}_{\Sigma_j}^{at},\min_{j}\mathfrak{O}_{\Sigma_j}^{pt}\right),\left(\min_{j}\mathcal{Y}_{\Sigma_j}^{at},\min_{j}\mathcal{Y}_{\Sigma_j}^{pt}\right)\right)$, thus
$$\Sigma_{CF}^{-}\leq CPFHPOG(\Sigma_{CF}^{1},\Sigma_{CF}^{2},\dots,\Sigma_{CF}^{w})\leq\Sigma_{CF}^{+}.$$

4 | MADM TECHNIQUE FOR PROPOSED METHOD

In this section, we discuss the MADM procedure based on the CPFHPA operator and CPFHPG operator to improve the worth of the derived operators.

For this, we consider the group of alternatives $\Sigma_{CF}^{1},\Sigma_{CF}^{2},\dots,\Sigma_{CF}^{w}$ with attributes $\Sigma_{AT}^{1},\Sigma_{AT}^{2},\dots,\Sigma_{AT}^{m}$, where all the attributes are considered for each alternative. Further, for all attributes we have the collection of weight vectors in the shape of priority degree $\frac{\overline{\overline{\Xi_J}}}{\sum_{j=1}^{w}\overline{\overline{\Xi_J}}}$. Additionally, we aim to compute the matrix by including their values in the shape of CPFSs, where, the truth, abstinence, and falsity grades are stated by: $\mathcal{Z}_{\Sigma_{CF}}(\mathfrak{g})=\left(\mathcal{Z}_{\Sigma_{CF}}^{at}(\mathfrak{g}),\mathcal{Z}_{\Sigma_{CF}}^{pt}(\mathfrak{g})\right),\mathfrak{O}_{\Sigma_{CF}}(\mathfrak{g})=\left(\mathfrak{O}_{\Sigma_{CF}}^{at}(\mathfrak{g}),\mathfrak{O}_{\Sigma_{CF}}^{pt}(\mathfrak{g})\right)$ and $\mathcal{Y}_{\Sigma_{CF}}(\mathfrak{g})=\left(\mathcal{Y}_{\Sigma_{CF}}^{at}(\mathfrak{g}),\mathcal{Y}_{\Sigma_{CF}}^{pt}(\mathfrak{g})\right)$ with a valuable condition, that is $0\leq\mathcal{Z}_{\Sigma_{CF}}^{at}(\mathfrak{g})+\mathfrak{O}_{\Sigma_{CF}}^{at}(\mathfrak{g})+\mathcal{Y}_{\Sigma_{CF}}^{at}(\mathfrak{g})\leq 1$ and $0\leq\mathcal{Z}_{\Sigma_{CF}}^{pt}(\mathfrak{g})+\mathfrak{O}_{\Sigma_{CF}}^{pt}(\mathfrak{g})+\mathcal{Y}_{\Sigma_{CF}}^{pt}(\mathfrak{g})\leq 1$. Moreover, we stated the refusal grade, such as $r_{\Sigma_{CF}}(\mathfrak{g})=\left(r_{\Sigma_{CF}}^{at}(\mathfrak{g}),r_{\Sigma_{CF}}^{pt}(\mathfrak{g})\right)=\left(1-\left(\mathcal{Z}_{\Sigma_{CF}}^{at}(\mathfrak{g})+\mathfrak{O}_{\Sigma_{CF}}^{at}(\mathfrak{g})+\mathcal{Y}_{\Sigma_{CF}}^{at}(\mathfrak{g})\right),1-\left(\mathcal{Z}_{\Sigma_{CF}}^{pt}(\mathfrak{g})+\mathfrak{O}_{\Sigma_{CF}}^{pt}(\mathfrak{g})+\mathcal{Y}_{\Sigma_{CF}}^{pt}(\mathfrak{g})\right)\right)$. Finally, we described the simple form of CPFN, such as: $\Sigma_{CF}^{j}=\left(\left(\mathcal{Z}_{\Sigma_j}^{at},\mathcal{Z}_{\Sigma_j}^{pt}\right),\left(\mathfrak{O}_{\Sigma_j}^{at},\mathfrak{O}_{\Sigma_j}^{pt}\right),\left(\mathcal{Y}_{\Sigma_j}^{at},\mathcal{Y}_{\Sigma_j}^{pt}\right)\right),j=1,2,\dots,w$. For addressing the above problem, we have the following procedure, such as:

Step 1: Collection of the decision matrix by including the CPFSs.

Step 2: Aggregate the decision matrix.

Step 3: Evaluate the score values.

Step 4: Rank all the alternatives and examine the best one.

To verify the above procedure, we evaluate some numerical examples for evaluating the best decision among the collection of finite alternatives.

4.1 Numerical example

The main contribution of this example is to talk about the solution of the best software company to open in the Asian market. For this, we consider the five best businesses in the Asian market represented by $\Sigma_{CF}^{1}, \Sigma_{CF}^{2}, \Sigma_{CF}^{3}, \Sigma_{CF}^{4}$ and Σ_{CF}^{5} represented different types of businesses. To select the best one, we will use the following features such as Σ_{AT}^{1}: growth analysis, Σ_{AT}^{2}: environmental impact, Σ_{AT}^{3}: social impact, and Σ_{AT}^{4}: political impact. Based on the above problem, we have the following procedure. For addressing the above problem, we have the following procedure, such as:

Step 1: Collection of the decision matrix by including the CPFSs, see Table 1.

Table 1. CPF decision matrix.

	Σ_{AT}^{1}	Σ_{AT}^{2}
Σ_{CF}^{1}	$((0.5,0.4),(0.3,0.2),(0.1,0.2))$	$((0.51,0.41),(0.31,0.21),(0.11,0.21))$
Σ_{CF}^{2}	$((0.4,0.3),(0.1,0.2),(0.2,0.1))$	$((0.41,0.31),(0.11,0.21),(0.21,0.11))$
Σ_{CF}^{3}	$((0.6,0.3),(0.2,0.2),(0.1,0.1))$	$((0.61,0.31),(0.21,0.21),(0.11,0.11))$
Σ_{CF}^{4}	$((0.7,0.5),(0.1,0.2),(0.1,0.2))$	$((0.71,0.51),(0.11,0.21),(0.11,0.21))$
Σ_{CF}^{5}	$((0.4,0.5),(0.2,0.2),(0.1,0.1))$	$((0.41,0.51),(0.21,0.21),(0.11,0.11))$
	Σ_{AT}^{3}	Σ_{AT}^{4}
Σ_{CF}^{1}	$((0.52,0.42),(0.32,0.22),(0.12,0.22))$	$((0.53,0.43),(0.33,0.23),(0.13,0.23))$
Σ_{CF}^{2}	$((0.42,0.32),(0.12,0.22),(0.22,0.12))$	$((0.43,0.33),(0.13,0.23),(0.23,0.13))$
Σ_{CF}^{3}	$((0.62,0.32),(0.22,0.22),(0.12,0.12))$	$((0.63,0.33),(0.23,0.23),(0.13,0.13))$
Σ_{CF}^{4}	$((0.72,0.52),(0.12,0.22),(0.12,0.22))$	$((0.73,0.53),(0.13,0.23),(0.13,0.23))$
Σ_{CF}^{5}	$((0.42,0.52),(0.22,0.22),(0.12,0.12))$	$((0.43,0.53),(0.23,0.23),(0.13,0.13))$

Step 2: Aggregate the decision matrix, see Table 2.

Table 2. Aggregated decision matrix.

	CPFHPA Operator	CPFHPG Operator
Σ_{CF}^{1}	$((0.501,0.4023),(0.301,0.2022),(0.101,0.2022))$	$((0.501,0.4023),(0.301,0.2023),(0.101,0.2023))$
Σ_{CF}^{2}	$((0.4037,0.3053),(0.1035,0.2052),(0.2036,0.105))$	$((0.4037,0.3052),(0.1037,0.2053),(0.2037,0.1053))$
Σ_{CF}^{3}	$((0.601,0.3053),(0.201,0.2052),(0.101,0.105))$	$((0.601,0.3052),(0.201,0.2053),(0.101,0.1053))$
Σ_{CF}^{4}	$((0.7011,0.501),(0.101,0.201),(0.101,0.201))$	$((0.701,0.501),(0.101,0.201),(0.101,0.201))$
Σ_{CF}^{5}	$((0.4037,0.5023),(0.2036,0.2022),(0.1035,0.1022))$	$((0.4037,0.5023),(0.2037,0.2023),(0.1037,0.1023))$

Step 3: Evaluate the score values, see Table 3.

Table 3. Representation of Score values.

	CPFHPA Operator	CPFHPG Operator
Σ_{CF}^{1}	0.0484	0.0483
Σ_{CF}^{2}	0.0458	0.0454
Σ_{CF}^{3}	0.1471	0.1468
Σ_{CF}^{4}	0.299	0.299
Σ_{CF}^{5}	0.1472	0.147

Step 4: Rank all the alternatives and examine the best one, see Table 4.

Table 4. Representation of ranking values.

Methods	Ranking values	Best one
CPFHPA Operator	$\Sigma_{CF}^{4} > \Sigma_{CF}^{5} > \Sigma_{CF}^{3} > \Sigma_{CF}^{1} > \Sigma_{CF}^{2}$	Σ_{CF}^{4}
CPFHPG Operator	$\Sigma_{CF}^{4} > \Sigma_{CF}^{5} > \Sigma_{CF}^{3} > \Sigma_{CF}^{1} > \Sigma_{CF}^{2}$	Σ_{CF}^{4}

The best option is Σ_{CF}^{4} based on two different operators, such as CPFHPA Operator and CPFHPG Operator. To further verify the above procedure, we aim to evaluate the comparison between proposed and existing operators to evaluate the importance and effectiveness of the presented operators.

5 | COMPARATIVE ANALYSIS

Operators In this section, we compare the exposed techniques by using the data in Table 1 with some prevailing operators to enhance the worth of the presented operators. For comparing the proposed techniques, we have the following existing techniques, such as Yager [11] derived the AOs for fuzzy information in 1994. Additionally, Xu [12] presented the AOs for IFSs. Xu and Yager [13] evaluated the geometric AOs for IFSs. Wang et al. [14] presented the geometric AOs for PFSs. Garg [15] examined the AOs for PFSs. Jana and Pal [16] evaluated the Hamacher operators for PFSs. Huang [17] presented the Hamacher operators for IFSs. Wei [18] exposed the Hamacher operators for PFSs. Yu [19] invented the PAOs for generalized IFSs. Finally, to use the data in Table 1, the comparative analysis in Table 5.

Table 5. Representation of comparative analysis.

Methods	Score values	Ranking values
Yager [11]	Failed	Failed
Xu [12]	Failed	Failed
Xu and Yager [13]	Failed	Failed
Wang et al. [14]	Failed	Failed
Garg [15]	Failed	Failed
Jana and Pal [16]	Failed	Failed
Huang [17]	Failed	Failed
Wei [18]	Failed	Failed
Yu [19]	Failed	Failed
Akram et al. [8]	0.0405,0.0407,0.1407,0.2907,0.1407	$\Sigma_{CF}^4 > \Sigma_{CF}^5 > \Sigma_{CF}^3 > \Sigma_{CF}^2 > \Sigma_{CF}^1$
	0.0399,0.0399,0.1399,0.2899,0.1399	$\Sigma_{CF}^4 > \Sigma_{CF}^5 > \Sigma_{CF}^3 > \Sigma_{CF}^1 > \Sigma_{CF}^2$
CPFHPA Operator	0.0484,0.0458,0.1471,0.299,0.1472	$\Sigma_{CF}^4 > \Sigma_{CF}^5 > \Sigma_{CF}^3 > \Sigma_{CF}^1 > \Sigma_{CF}^2$
CPFHPG Operator	0.0483,0.0454,0.1468,0.299,0.147	$\Sigma_{CF}^4 > \Sigma_{CF}^5 > \Sigma_{CF}^3 > \Sigma_{CF}^1 > \Sigma_{CF}^2$

The best option is Σ_{CF}^4 based on two different operators, such as CPFHPA Operator, CPFHPG Operator, and Hamacher aggregation operators for CPFSs, proposed by Akram et al. [8]. The existing techniques have failed because of many restrictions and limitations.

6 | CONCLUSION

The major contribution of this paper is listed below:

1. We presented the CIFHPA operator, CIFHPOA operator, CIFHPG operator, and CIFHPOG operator. Furthermore, some remarkable properties of these operators are also examined.
2. We contributed to the improvement of the MADM procedure with a new plan of an algorithm in the CPF environment.
3. We selected and solved a MADM problem that evaluates the best optimal between the collection of decisions based on derived operators.
4. We checked the reliability and effectiveness of these operators with the help of a comparative analysis between the proposed techniques and some prevailing operators.

In the upcoming times, we aim to convert the idea of T-spherical fuzzy sets [20] into complex T-spherical fuzzy sets and try to discuss their application in the field of similarity measures [21] and aggregation operators [22] for the proposed operators to enhance the worth of the invented theory.

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