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Heronian Mean Operators Based Multi-Attribute Decision Making Algorithm Using T-Spherical Fuzzy Information

Akhtar Ali[1], Darko Božanić[2], Maria Akram[3], Saba Ijaz[4]

Authors' Affiliation:

[1,3,4]Department of Mathematics, Riphah Institute of Computing & Applied Sciences, Riphah International University (Lahore Campus), Lahore, Pakistan.
[2] Military Academy, The University of Defence in Belgrade, Belgrade, Serbia.

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Corresponding author(s):

Names: Darko Božanić[2]
Email: dbozanic@yahoo.com

ABSTRACT

To cope with human opinion in real-life problems with the help of the truth degree (TD), abstinence degree (AD), falsity degree (FD), and refusal degree (RD), the notion of T-spherical fuzzy set (TSFS) is developed as a replacement of the picture fuzzy set (PFS) which fails to handle information due to the strict condition on the TD, AD, FD, and RD. TSFS allows any value for TD, AD, FD, and RD from the $[0,1]$ with a parameter $p \in \mathbb{Z}^{+}$. Several aggregation operators (AOs) exist for the aggregation of the information among which the Heronian mean (HM) operator (HMO) is the significant one. This manuscript aims to discuss the theory of HMOs in a T-spherical fuzzy (TSF) environment that will allow us to model human opinion based on complex information in several practical scenarios. We develop the generalized TSF (GTSFHM) operator, generalized TSF weighted HM (GTSFWHM) operator, TSF geometric HM (TSFGHM) operator and TSF weighted geometric HM (TSFWGHM) operator. Further, the basic properties of the aggregation of the HMOs in the TSF environment are also discussed. Moreover, an enterprise resource planning problem based on the multi-attribute decision-making (MADM) procedure is investigated given the TSF HMOs followed by a comprehensive example showing the applicability of the proposed HMOs. Some exceptional cases of the proposed HMOs are investigated and a comparison of the newly developed TSF HMOs with previously developed HMOs is examined where the advantages of the new HMOs are discussed.

Keywords: Heronian mean operators; T-Spherical fuzzy sets. Spherical fuzzy sets; Multi-attribute decision-making.

1 | INTRODUCTION

MADM is a capable technique that can give the ranking of the outcomes for the limited options as per the quality of the alternatives among various other options, and it is one of the significant techniques of decision-making sciences. In a genuine decision-making procedure, a significant issue is a way to communicate the quality worth even more proficiently and precisely. Because of the intricacy of decision-making issues and the fuzziness of decision-making conditions, it is not sufficient to communicate the characteristics of the alternatives precisely. For this, Zadeh [1] demonstrated the fuzzy set (FS) theory, which gave the idea of truth degree (TD) of vague concepts like age, intelligence, beauty, etc. The TD is a value from $[0,1]$ associated with an element by a membership function. Atanassov [2] observed that describing the TD of elements only is not sufficient and there exists a falsity degree (FD) along with TD and the notion of intuitionistic FS (IFS) is developed. The TD and FD in IFS are such that their sum lies in $[0,1]$. Several scholars have utilized the theory of IFSs in many fields [3]–[7]. In many problems, the theory of IFS becomes inapplicable especially when we have duplets like $(0.9, 0.4)$ where the sum of both TD and FD exceeds 1 i.e., $sum(0.9,0.4) = 1.3$. Hence many such duplets cannot be considered intuitionistic. To cover this limitation, Yager [8] introduced the idea of Pythagorean FS (PyFS) which increases the range of Atanassov IFS. PyFS is also based on the TD and FD with a condition that $sum\ (TD^2, FD^2)$ does not exceed 1. For example, in the duplet $(0.9, 0.4)$ $sum(0.9^2, 0.4^2) = 0.97 \in [0,1]$ which categorizes such duplet as Pythagorean fuzzy. Several scholars have utilized the theory of PyFSs in many fields [9]–[13]. From this example, it is clear that Yager [8] improved the limitation that is faced by Atanassov IFS but still there exists some pairs that cannot be categorized as Pythagorean or intuitionistic. For example, the pair $(0.6, 0.9)$ does not represent intuitionistic or Pythagorean fuzzy information as $sum(0.6,0.9) = 1.5 \notin [0,1]$ and $sum(0.6^2, 0.9^2) = 1.17 \notin [0,1]$. Due to this, Yager [14] developed the idea of q-rung ortho pair FS (q-ROFS) which can handle any pair with variable $p \in \mathbb{Z}^+$. Numerous scholars have utilized the theory of q-ROFSs in many fields [15]–[19].

In 2013, Cuong [20] suggested that the human opinion does not have only two aspects as described by the TD and FD of IFS, PyFS, and q-ROFS but it has an abstinence degree (AD) and refusal degree (RD) which was not discussed in the previous fuzzy frameworks. Cuong introduced the concept of a picture fuzzy set (PFS) that is described by TD, AD, FD, and RD with a condition that the sum of TD, AD, and FD must be lei between $[0,1]$. The frame of PFS offered more accuracy than previous fuzzy frameworks and hence greatly utilized by several scholars [21]–[25]. Mahmood et al. [26] enhanced the PFS and gave the notion of spherical FS (SFS) and TSFS by providing more independence for the selection of TD, AD, and FD. In TSFSs, the TD, AD, and FD are associated with a parameter p with the condition that $sum\ (TD^p, AD^p, FD^p)$ lies between $[0,1]$ for $p \in \mathbb{Z}^+$. Due to the greater flexibility of TSFS, some major work on the theory and applications of the relations and aggregation of TSFSs, and information measures of TSFSs are investigated in [27]–[34].

Several AOs are introduced to deal with MADM problems which are based on some t-norms and t-conorms including Einstein aggregation operators [35], [36] based on Einstein t-norms, averaging, and geometric AOs [37], [38] based on algebraic sum and product and so on. HMO is one of the widely used aggregation operators and it was first initiated by [39] and becomes a hot research topic discussed by several authors in several fuzzy frameworks. Yu [40] and Li and Wei [41] developed HMOs in intuitionistic and Pythagorean fuzzy environments respectively. Picture fuzzy HMOs for MADM have been studied by Wei et al. [42] while hesitant fuzzy HMOs are developed by Yu [43]. Some other extensive work on HMOs and their applications can be found in [44]–[47]. In a previous literature review, we noticed that most experts have faced the following three major dilemmas:

1. How do we obtain a new aggregation operator in the availability of TSFSs?

2. How we aggregate the collection of T-spherical fuzzy numbers.
3. How do we illustrate the ranking orders of the aggregated values?

Hence, the main theme of this analysis is to evaluate the fundamental theory of Heronian mean operators for TSFSs and discuss their valuable results and properties. The studies established on HMOs in [16], [40], [41] discussed only the TD and FD of the vague information and does not provide any knowledge about the AD and RD of the information. Moreover, the HMOs that are established in the picture fuzzy environment discussed the TD, AD, FD, and RD of the information but with lesser flexibility. Therefore, this paper aims to develop the HMOs in a T-spherical fuzzy (TSF) environment that will overcome the limitations of the HMOs of IFSs, PyFSs, q-ROFSs, and PFSs with more flexibility and less information loss. Further, HMO is very beneficial and most dynamic for evaluating awkward and unreliable information in realistic dilemmas. Moreover, the theory of HMO is the modified form of aggregation/geometric aggregation operators. The main mathematical shape of the Heronian mean is computed in the shape: $H = \frac{1}{3}\left(a + b + (ab)^{\frac{1}{2}}\right)$, where a, b represented real numbers. A lot of scholars have employed the theory of HMOs in the environment of different fields. Keeping the advantages of the HMOs and using a valuable technique of TSFS, we discuss the main influence of this theory in the shape of the below points:

1. To obtain a new aggregation operator in the availability of TSFSs, we evaluate the theory of GTSFHM, GTSFWHM, TSFGHM, and TSFWGHM operators and examine their valuable results and properties.
2. To aggregate the collection of T-spherical fuzzy numbers, we use the theory of GTSFHM, GTSFWHM, TSFGHM, and TSFWGHM operators to improve the worth of the diagnosed operators.
3. To illustrate the ranking orders of the aggregated values, a MADM procedure is developed by using the investigated operators with the help of TSF information.
4. The comparative survey and the benefits of the proposed results are illustrated numerically with the help of some graphical demonstration.

The structure of this manuscript is organized in the following ways: In section 2, we recall the ideas of TSFS and related terms. In section 3, HMOs are developed in a TSF environment and the notion of GTSFHM, GTSFWHM, TSFGHM, and TSFWGHM operators are discussed in Section 4. In section 5, some special cases of the TSF HMOs are discussed because of some restrictions. In section 6, MADM algorithms based on HMOs using TSF information are set up and supported by a numerical example. Section 7 is based on the comparative study of proposed and previous HMOs where the advantages of the TSFHM operators are studied. In section 8, we summarize the paper and discuss some future directions.

2 | PRELIMINARIES

This section aims to discuss the relevant previous literature and to discuss the gap between existing and previous literature that leads us to develop the idea of HMOs in a T-spherical fuzzy environment.

Throughout this paper, X denotes a non-empty set while s, h, d, and r denote the TD, AD, FD, and RD of an element. Further, $\mathfrak{i}, \mathfrak{j} \in I = \{1, 2, 3, \ldots\}$.

Definition 1: [26] A TSFS is having the shape:

$$T = \{(s(x), h(x), d(x)) : x \in X\} \tag{1}$$

where $s(x), h(x)$, and $d(x)$ are mappings from X to $[0, 1]$ with a restriction that $0 \leq s^p(x) + h^p(x) + d^p(x) \leq 1$ for some least $p \in \mathbb{Z}^+$. The RD is defined by $r(x) = \left(1 - \left(s^p(x) + h^p(x) + d^p(x)\right)\right)^{\frac{1}{p}}$ and the triplet $(s(x), h(x), d(x))$ is a T-spherical fuzzy number (TSFN). Further, for TSFNs $T_1 = (s_1(x), h_1(x), d_1(x))$ and $T_2 = (s_2(x), h_2(x), d_2(x))$ and $\delta \geq 1$

$$T_1 \oplus T_2 = \left(\left(s_1^p(x) + s_2^p(x) - s_1^p(x) s_2^p(x)\right)^{\frac{1}{p}}, h_1(x) h_2(x), d_1(x) d_2(x)\right) \tag{2}$$

$$T_1 \otimes T_2 = \left(s_1(x) s_2(x), \left(h_1^p(x) + h_2^p(x) - h_1^p(x) h_2^p(x)\right)^{\frac{1}{p}}, \left(d_1^p(x) + d_2^p(x) - d_1^p(x) d_2^p(x)\right)^{\frac{1}{p}}\right) \tag{3}$$

$$\delta T_1 = \left(\left(1 - \left(1 - s_1^p(x)\right)^{\delta}\right)^{\frac{1}{p}}, h_1^{\delta}(x), d_1^{\delta}(x)\right) \tag{4}$$

$$T_1^{\delta} = \left(s_1^{\delta}(x), \left(1 - \left(1 - h_1^p(x)\right)^{\delta}\right)^{\frac{1}{p}}, \left(1 - \left(1 - d_1^p(x)\right)^{\delta}\right)^{\frac{1}{p}}\right) \tag{5}$$

Definition 2: [26] Let T be a TSFN. Then the score value of T is defined as:

$$SC(T) = s^p(\varkappa) - d^p(\varkappa) \tag{6}$$

where $SC(T) \in [-1, 1]$, in case, when score function could not differentiate between two TSFNs. Then the concept of accuracy function is used.

Definition 3: [26] The accuracy function of a TSFN T is defined as:

$$AC(T) = s^p(\varkappa) + h^p(\varkappa) + d^p(\varkappa), AC(T) \in [0, 1] \tag{7}$$

By the definition of the score function of TSFNs, for two TSFNs T_1 and T_2, we have

1. T_1 is superior to T_2 if $SC(T_1) > SC(T_2)$.
2. T_1 is inferior to T_2 if $SC(T_1) < SC(T_2)$.
3. If $SC(T_1) = SC(T_2)$ for two TSFNs. Then
    1) T_1 is superior to T_2 if $AC(T_1) > AC(T_2)$.
    2) T_1 is inferior to T_2 if $AC(T_1) < AC(T_2)$.
    3) T_1 is like T_2 if $AC(T_1) = AC(T_2)$

Definition 4: [16] Let $T_{\mathfrak{i}}$ be a collection of nonnegative numbers. Then HM operator is defined as:

$$HM(T_1, T_2, \dots, T_{\underline{n}}) = \frac{2}{\underline{n}(\underline{n}+1)} \bigoplus_{i=1}^{\underline{n}} \bigoplus_{j=i}^{\underline{n}} \sqrt{T_i T_j} \tag{8}$$

Definition 5: [16] Let $\alpha, \beta > 0$, and T_i be a collection of nonnegative numbers. Then generalized HM (GHM) operator is defined as:

$$GHM(T_1, T_2, \dots, T_{\underline{n}}) = \left(\frac{2}{\underline{n}(\underline{n}+1)} \bigoplus_{i=1}^{\underline{n}} \bigoplus_{j=i}^{\underline{n}} T_i^{\alpha} \otimes T_j^{\beta}\right)^{\frac{1}{\alpha+\beta}} \tag{9}$$

In literature, the theory of HMOs has been greatly discussed in different fuzzy settings for MADM purposes. Some potential work in this regard is discussed in [16], [39]–[43], [48]. Where some work failed because of the limitation in assigning the membership grades e.g., HMOs of the IFS, PyFS, PFS, and SFS while some leads to loss of information e.g., HMOs of the IFS, PyFS, and QROFS, etc. Due to this reason, we aim to develop the theory of HMOs in the setting TSFSs and try to investigate their applicability.

3 | GENERALIZED T-SPHERICAL FUZZY HERONIAN MEAN OPERATORS

In this section, we define the notions of the GTSFHM operator and GTSFWHM operator. The essential properties of aggregation are proved for the said HMOs. From this section onward $w_i = (w_1, w_2, w_3, \dots, w_{\underline{n}})$ denotes the weight vector of T_i, where w_i represents the degree of importance of T_i, such that $w_i > 0$ and $\sum_{i=1}^{\underline{n}} w_i = 1$.

Definition 6: Let $T_i = (s_i, h_i, d_i)$ be some family of TSFNs and $\alpha, \beta \geq 0$. Then GTSFHM operator is a function $T^{\underline{n}} \to T$ defined as:

$$GTSFHM\ (T_1, T_2, \dots, T_{\underline{n}}) = \left(\frac{2}{\underline{n}(\underline{n}+1)} \bigoplus_{i=1}^{\underline{n}} \bigoplus_{j=i}^{\underline{n}} \left(T_i^{\alpha} \otimes T_j^{\beta}\right)\right)^{\frac{1}{\alpha+\beta}} \tag{10}$$

Theorem 1: Let $T_i = (s_i, h_i, d_i)$ be some family of TSFNs, and $\alpha, \beta > 0$. Then GTSFHM operator results in the form of a TSFN given by:

$$GTSFHM\ (T_1, T_2, \dots, T_{\underline{n}}) = \left(\begin{array}{c} \left(\sqrt[p]{1 - \prod_{i=1, j=i}^{\underline{n}} \left(1 - s_i^{\alpha p} s_j^{\beta p}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}}\right)^{\frac{1}{\alpha+\beta}}, \\ \sqrt[p]{1 - \left(1 - \prod_{i=1, j=i}^{\underline{n}} \left(1 - (1 - h_i^p)^{\alpha} (1 - h_j^p)^{\beta}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}\right)^{\frac{1}{\alpha+\beta}}}, \\ \sqrt[p]{1 - \left(1 - \prod_{i=1, j=i}^{\underline{n}} \left(1 - (1 - d_i^p)^{\alpha} (1 - d_j^p)^{\beta}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}\right)^{\frac{1}{\alpha+\beta}}} \end{array}\right) \tag{11}$$

Noticed that the theory in Equation 11 is the generalized form of the Heronian mean operators based on FSs, IFSs, PyFSs, PFSs, and SFSs. Further, the proposed work in Equation 11 is also the modified version of the averaging and geometric aggregation operators for the theory of FSs, IFSs, PyFSs, PFSs, and SFSs.

Proof: By the Definition 1,

$$T_{i}^{\alpha}=\left(s_{i}^{\alpha},\sqrt[p]{1-\left(1-h_{i}^{p}\right)^{\alpha}},\sqrt[p]{1-\left(1-d_{i}^{p}\right)^{\alpha}}\right)$$
$$T_{j}^{\beta}=\left(s_{j}^{\beta},\sqrt[p]{1-\left(1-h_{j}^{p}\right)^{\beta}},\sqrt[p]{1-\left(1-d_{j}^{p}\right)^{\beta}}\right)$$
$$T_{i}^{\alpha}\otimes T_{j}^{\beta}=\left(s_{i}^{\alpha}s_{j}^{\beta},\sqrt[p]{1-\left(1-h_{i}^{p}\right)^{\alpha}\left(1-h_{j}^{p}\right)^{\beta}},\sqrt[p]{1-\left(1-d_{i}^{p}\right)^{\alpha}\left(1-d_{j}^{p}\right)^{\beta}}\right)$$

Therefore,

$$\bigoplus_{i=1}^{n}\bigoplus_{j=i}^{n}\left(T_{i}^{\alpha}\otimes T_{j}^{\beta}\right)=\left(\sqrt[p]{1-\prod_{i=1,j=i}^{n}\left(1-s_{i}^{\alpha p}s_{j}^{\beta p}\right)},\sqrt[p]{\prod_{i=1,j=i}^{n}\left(1-\left(1-h_{i}^{p}\right)^{\alpha}\left(1-h_{j}^{p}\right)^{\beta}\right)},\sqrt[p]{\prod_{i=1,j=i}^{n}\left(1-\left(1-d_{i}^{p}\right)^{\alpha}\left(1-d_{j}^{p}\right)^{\beta}\right)}\right)$$

Furthermore,

$$\frac{2}{n(n+1)}\bigoplus_{i=1}^{n}\bigoplus_{j=i}^{n}\left(T_{i}^{\alpha}\otimes T_{j}^{\beta}\right)=\left(\begin{array}{c}\sqrt[p]{1-\prod_{i=1,j=i}^{n}\left(1-s_{i}^{\alpha p}s_{j}^{\beta p}\right)^{\frac{2}{n(n+1)}}},\\ \sqrt[p]{\prod_{i=1,j=i}^{n}\left(1-\left(1-h_{i}^{p}\right)^{\alpha}\left(1-h_{j}^{p}\right)^{\beta}\right)^{\frac{2}{n(n+1)}}},\\ \sqrt[p]{\prod_{i=1,j=i}^{n}\left(1-\left(1-d_{i}^{p}\right)^{\alpha}\left(1-d_{j}^{p}\right)^{\beta}\right)^{\frac{2}{n(n+1)}}}\end{array}\right),$$

$$\left(\frac{2}{n(n+1)}\bigoplus_{i=1}^{n}\bigoplus_{j=i}^{n}\left(T_{i}^{\alpha}\otimes T_{j}^{\beta}\right)\right)^{\frac{1}{\alpha+\beta}}=\left(\begin{array}{c}\left(\sqrt[p]{1-\prod_{i=1,j=i}^{n}\left(1-s_{i}^{\alpha p}s_{j}^{\beta p}\right)^{\frac{2}{n(n+1)}}}\right)^{\frac{1}{\alpha+\beta}},\\ \sqrt[p]{1-\left(1-\prod_{i=1,j=i}^{n}\left(1-\left(1-h_{i}^{p}\right)^{\alpha}\left(1-h_{j}^{p}\right)^{\beta}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{\alpha+\beta}}},\\ \sqrt[p]{1-\left(1-\prod_{i=1,j=i}^{n}\left(1-\left(1-d_{i}^{p}\right)^{\alpha}\left(1-d_{j}^{p}\right)^{\beta}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{\alpha+\beta}}}\end{array}\right)$$

Therefore,

$$GTSFHM\left(T_1,T_2,\ldots,T_{\underline{n}}\right)=\left(\begin{array}{c}\left(\sqrt[p]{1-\prod_{i=1,j=i}^{\underline{n}}\left(1-s_i^{\alpha p}s_j^{\beta p}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}}\right)^{\frac{1}{\alpha+\beta}},\\ \sqrt[p]{1-\left(1-\prod_{i=1,j=i}^{\underline{n}}\left(1-\left(1-h_i^p\right)^{\alpha}\left(1-h_j^p\right)^{\beta}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}\right)^{\frac{1}{\alpha+\beta}}},\\ \sqrt[p]{1-\left(1-\prod_{i=1,j=i}^{\underline{n}}\left(1-\left(1-d_i^p\right)^{\alpha}\left(1-d_j^p\right)^{\beta}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}\right)^{\frac{1}{\alpha+\beta}}}\end{array}\right)$$

Hence proved.

We can easily show that GTSFHM operators satisfy the properties given below.

1. **Idempotency:** If T_i all are equal, as $T_i = T$ for all i, then.

$$GTSFHM\left(T_1,\ T_2,T_3,\ldots,T_{\underline{n}}\right)=T \tag{12}$$

2. **Boundedness:** Suppose that T_i be some family of TSFNs, and suppose

$$T^{-}=\min_{i} T_i,T^{+}=\max_{i} T_i$$

Then

$$T^{-}\le GTSFHM\left(T_1,\ T_2,T_3,\ldots,T_{\underline{n}}\right)\le T^{+} \tag{13}$$

3. **Monotonicity:** Suppose T_i and $\acute{T}_i$ be the two families of TSFNs, if $T_i \le\ \acute{T}_i$.Then

$$GTSFHM\left(T_1,\ T_2,T_3,\ldots,T_{\underline{n}}\right)\le GTSFHM\left(\acute{T}_1,\acute{T}_2,\acute{T}_3,\ldots,\acute{T}_{\underline{n}}\right) \tag{14}$$

By Definition 6, we can see that the GTSFHM operator does not reflect the weights of the information. In many situations, particularly in MADM, the weights of characteristics (attributes) participate a key role in the procedure of the aggregation. Therefore, we introduce the GTSFWHM operator as given below.

Definition 7: Let $T_i=(s_i,h_i,d_i)$ be some family of TSFNs, and $\alpha,\beta>0$. Then GTSFWHM operator is a function $T^{\underline{n}}\to T$ defined as:

$$\mathrm{GT}SFWHM\left(T_1,T_2,T_3,\ldots,T_{\underline{n}}\right)=\left(\bigoplus_{i=1}^{\underline{n}}\bigoplus_{j=i}^{\underline{n}}\left(w_iw_j(T_i)^{\alpha}\otimes\left(T_j\right)^{\beta}\right)\right)^{\frac{1}{\alpha+\beta}} \tag{15}$$

Based on Equation 15 and Definition 1, we have the following theorem.

Theorem 2: Let $T_i=(s_i,h_i,d_i)$ be some family of TSFNs, and $\alpha,\beta>0$. Then GTSFWHM operator results in the form of a TSFN given by:

$$GTSFWHM\left(T_1, T_2, T_3, \dots, T_{\underline{n}}\right) = \left(\begin{array}{c} \left(\sqrt[p]{1 - \prod_{\imath=1, \jmath=\imath}^{\underline{n}} \left(1 - s_{\imath}^{\alpha p} s_{\jmath}^{\beta p}\right)^{w_{\imath} w_{\jmath}}} \right)^{\frac{1}{\alpha+\beta}}, \\ \sqrt[p]{1 - \left(1 - \prod_{\imath=1, \jmath=\imath}^{\underline{n}} \left(1 - \left(1 - h_{\imath}^{p}\right)^{\alpha} \left(1 - h_{\jmath}^{p}\right)^{\beta}\right)^{w_{\imath} w_{\jmath}}\right)^{\frac{1}{\alpha+\beta}}}, \\ \sqrt[p]{1 - \left(1 - \prod_{\imath=1, \jmath=\imath}^{\underline{n}} \left(1 - \left(1 - d_{\imath}^{p}\right)^{\alpha} \left(1 - d_{\jmath}^{p}\right)^{\beta}\right)^{w_{\imath} w_{\jmath}}\right)^{\frac{1}{\alpha+\beta}}} \end{array} \right) \tag{16}$$

Noticed that the theory in Equation 16 is the generalized form of the weighted Heronian mean operators based on FSs, IFSs, PyFSs, PFSs, and SFSs. Further, the proposed work in Equation 16 is also the modified version of the weighted averaging and geometric aggregation operators for the theory of FSs, IFSs, PyFSs, PFSs, and SFSs.

Proof: By Definition 1,

$$T_{\imath}^{\alpha} = \left(s_{\imath}^{\alpha}, \sqrt[p]{1 - \left(1 - h_{\imath}^{p}\right)^{\alpha}}, \sqrt[p]{1 - \left(1 - h_{\jmath}^{p}\right)^{\alpha}}\right), T_{\jmath}^{\beta} = \left(s_{\jmath}^{\beta}, \sqrt[p]{1 - \left(1 - h_{\jmath}^{p}\right)^{\beta}}, \sqrt[p]{1 - \left(1 - d_{\jmath}^{p}\right)^{\beta}}\right)$$

Thus,

$$T_{\imath}^{\alpha} \otimes T_{\jmath}^{\beta} = \left(s_{\imath}^{\alpha} s_{\jmath}^{\beta}, \sqrt[p]{1 - \left(1 - h_{\imath}^{p}\right)^{\alpha} \left(1 - h_{\jmath}^{p}\right)^{\beta}}, \sqrt[p]{1 - \left(1 - d_{\imath}^{p}\right)^{\alpha} \left(1 - d_{\jmath}^{p}\right)^{\beta}}\right)$$

Therefore,

$$w_{\imath} w_{\jmath} \left(T_{\imath}^{\alpha} \otimes T_{\jmath}^{\beta}\right)$$
$$= \left(\sqrt[p]{1 - \left(1 - s_{\imath}^{\alpha p} s_{\jmath}^{\beta p}\right)^{w_{\imath} w_{\jmath}}}, \left(\sqrt[p]{1 - \left(1 - h_{\imath}^{p}\right)^{\alpha} \left(1 - h_{\jmath}^{p}\right)^{\beta}}\right)^{w_{\imath} w_{\jmath}}, \left(\sqrt[p]{1 - \left(1 - d_{\imath}^{p}\right)^{\alpha} \left(1 - d_{\jmath}^{p}\right)^{\beta}}\right)^{w_{\imath} w_{\jmath}}\right)$$

$$\bigoplus_{\imath=1}^{\underline{n}} \bigoplus_{\jmath=\imath}^{\underline{n}} w_{\imath} w_{\jmath} \left(T_{\imath}^{\alpha} \otimes T_{\jmath}^{\beta}\right) = \left(\begin{array}{c} \sqrt[p]{1 - \prod_{\imath=1, \jmath=\imath}^{\underline{n}} \left(1 - s_{\imath}^{\alpha p} s_{\jmath}^{\beta p}\right)^{w_{\imath} w_{\jmath}}}, \\ \left(\sqrt[p]{\prod_{\imath=1, \jmath=\imath}^{\underline{n}} \left(1 - \left(1 - h_{\imath}^{p}\right)^{\alpha} \left(1 - h_{\jmath}^{p}\right)^{\beta}\right)}\right)^{w_{\imath} w_{\jmath}} \\ , \left(\sqrt[p]{\prod_{\imath=1, \jmath=\imath}^{\underline{n}} \left(1 - \left(1 - d_{\imath}^{p}\right)^{\alpha} \left(1 - d_{\jmath}^{p}\right)^{\beta}\right)}\right)^{w_{\imath} w_{\jmath}} \end{array} \right)$$

$$\left(\bigoplus_{i=1}^{n}\bigoplus_{j=1}^{n} w_i w_j \left(T_i^{\alpha} \otimes T_j^{\beta}\right)\right)^{\frac{1}{\alpha+\beta}} = \begin{pmatrix} \left(\sqrt[p]{1-\prod_{i=1,j=i}^{n}\left(1-s_i^{\alpha p} s_j^{\beta p}\right)^{w_i w_j}}\right)^{\frac{1}{\alpha+\beta}}, \\ \sqrt[p]{1-\left(1-\prod_{i=1,j=i}^{n}\left(1-\left(1-h_i^{p}\right)^{\alpha}\left(1-h_j^{p}\right)^{\beta}\right)^{w_i w_j}\right)^{\frac{1}{\alpha+\beta}}}, \\ \sqrt[p]{1-\left(1-\prod_{i=1,j=i}^{n}\left(1-\left(1-d_i^{p}\right)^{\alpha}\left(1-d_j^{p}\right)^{\beta}\right)^{w_i w_j}\right)^{\frac{1}{\alpha+\beta}}} \end{pmatrix}$$

Hence,

$$GTSFWHM(T_1, T_2, T_3, \dots, T_n) = \begin{pmatrix} \left(\sqrt[p]{1-\prod_{i=1,j=i}^{n}\left(1-s_i^{\alpha p} s_j^{\beta p}\right)^{w_i w_j}}\right)^{\frac{1}{\alpha+\beta}}, \\ \sqrt[p]{1-\left(1-\prod_{i=1,j=i}^{n}\left(1-\left(1-h_i^{p}\right)^{\alpha}\left(1-h_j^{p}\right)^{\beta}\right)^{w_i w_j}\right)^{\frac{1}{\alpha+\beta}}}, \\ \sqrt[p]{1-\left(1-\prod_{i=1,j=i}^{n}\left(1-\left(1-d_i^{p}\right)^{\alpha}\left(1-d_j^{p}\right)^{\beta}\right)^{w_i w_j}\right)^{\frac{1}{\alpha+\beta}}} \end{pmatrix}$$

Hence Proved

We can easily show that GTSFWHM operators satisfy the properties given below.

1. **Idempotency:** If all T_i are equal, as $T_i = T$ for all i, then:

$$GTSFWHM(T_1, T_2, T_3, \dots, T_n) = T \tag{17}$$

2. **Boundedness:** Suppose that T_i be some family of TSFNs, and suppose

$$T^{-} = \min_{i} T_i, T^{+} = \max_{i} T_i$$

Then

$$T^{-} \leq GTSFWHM(T_1, T_2, T_3, \dots, T_n) \leq T^{+} \tag{18}$$

3. **Monotonicity:** Suppose T_i and $\acute{T}_i$ be some two families of TSFNs. If $T_i \leq \acute{T}_i$, for all i, then

$$GTSFWHM(T_1, T_2, T_3, \dots, T_n) \leq GTSFWHM(\acute{T}_1, \acute{T}_2, \acute{T}_3, \dots, \acute{T}_n) \tag{19}$$

An example to illustrate the applicability of the GTSFWHM operator is given as follows:

Example 1: Let $T_1 = (0.4, 0.6, 0.5)$, $T_2 = (0.7, 0.5, 0.8)$, $T_3 = (0.3, 0.5, 0.3)$ and $T_4 = (0.4, 0.8, 0.4)$ be four TSFNs for $p = 3$ and also let $\alpha = \beta = 1$ and $w = (0.25, 0.35, 0.27, 0.13)$. Then

$$GTSFWHM(T_1, T_2, T_3, T_4)$$

$$= \left(\begin{array}{c} \left(\sqrt[3]{1 - \{(1 - 0.4^3 0.4^3)^{0.25^2}(1 - 0.7^3 0.7^3)^{0.35^2}(1 - 0.3^3 0.3^3)^{0.27^2}(1 - 0.4^3 0.4^3)^{0.13^2}\}} \right)^{\frac{1}{2}}, \\ \sqrt[3]{1 - \begin{pmatrix} 1 - (1 - (1 - 0.6^3)^1(1 - 0.6^3)^1)^{0.25^2}(1 - (1 - 0.5^3)^1(1 - 0.5^3)^1)^{0.35^2} \\ (1 - (1 - 0.5^3)^1(1 - 0.5^3)^1)^{0.27^2}(1 - (1 - 0.8^3)^1(1 - 0.8^3)^1)^{0.13^2} \end{pmatrix}^{\frac{1}{2}}}, \\ \sqrt[3]{1 - \begin{pmatrix} 1 - (1 - (1 - 0.5^3)^1(1 - 0.5^3)^1)^{0.25^2}(1 - (1 - 0.8^3)^1(1 - 0.8^3)^1)^{0.35^2} \\ (1 - (1 - 0.3^3)^1(1 - 0.3^3)^1)^{0.27^2}(1 - (1 - 0.4^3)^1(1 - 0.4^3)^1)^{0.13^2} \end{pmatrix}^{\frac{1}{2}}} \end{array} \right)$$

$$GTSFWHM(T_1, T_2, T_3, T_4) = (0.602651, 0.999127, 0.999568)$$

4 | RESULTS T-SPHERICAL FUZZY GEOMETRIC HERONIAN MEAN OPERATORS

In this section, we introduce the TSFGHM operator and TSFWGHM operator. Some properties of these AOs are also investigated. An example to show the use of the TSFGHM operator is also studied.

Definition 8: Let $T_i = (s_i, h_i, d_i)$ be some family of TSFNs, and $\alpha, \beta \geq 0$. The TSFGHM operator is a function $T^{\underline{n}} \to T$ defined as:

$$TSFGHM\left(T_1, T_2, \ldots, T_{\underline{n}}\right) = \frac{1}{\alpha + \beta}\left(\bigotimes_{i=1}^{\underline{n}}\bigotimes_{j=i}^{\underline{n}}\left(\alpha T_i \oplus \beta T_j\right)\right)^{\frac{2}{\underline{n}(\underline{n}+1)}} \tag{20}$$

Based on Equation 20 and Definition 1, we have the following theorem.

Theorem 3: Let $T_i = (s_i, h_i, d_i)$ be some family of TSFNs, and $\alpha, \beta > 0$. Then TSFGHM operator results in the form of a TSFN given by:

$$TSFGHM(T_1, T_2, \ldots, T_{\underline{n}}) = \left(\begin{array}{c} \left(\sqrt[p]{1 - \prod_{i=1, j=i}^{\underline{n}} \left(1 - s_i^{\alpha p} s_j^{\beta p}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}} \right)^{\frac{1}{\alpha+\beta}}, \\ \sqrt[p]{1 - \left(1 - \prod_{i=1, j=i}^{\underline{n}} \left(1 - (1 - h_i^p)^{\alpha}(1 - h_j^p)^{\beta}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}\right)^{\frac{1}{\alpha+\beta}}}, \\ \sqrt[p]{1 - \left(1 - \prod_{i=1, j=i}^{\underline{n}} \left(1 - (1 - d_i^p)^{\alpha}(1 - d_j^p)^{\beta}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}\right)^{\frac{1}{\alpha+\beta}}} \end{array} \right) \tag{21}$$

Noticed that the theory in Equation 21 is the generalized form of the geometric Heronian mean operators based on FSs, IFSs, PyFSs, PFSs, and SFSs. Further, the proposed work in Equation 21 is also the modified version of the averaging and geometric aggregation operators for the theory of FSs, IFSs, PyFSs, PFSs, and SFSs.

Proof: By Definition 1,

$$\alpha T_{\imath} = \left(\sqrt[p]{1-\left(1-s_{\imath}^{p}\right)^{\alpha}}, h_{\imath}^{\alpha}, d_{\imath}^{\alpha}\right), \ \beta T_{\jmath} = \left(\sqrt[p]{1-\left(1-s_{\jmath}^{p}\right)^{\beta}}, h_{\jmath}^{\beta}, d_{\jmath}^{\beta}\right)$$

Thus,

$$\alpha T_{\imath} \oplus \beta T_{\jmath} = \left(\sqrt[p]{1-\left(1-s_{\imath}^{p}\right)^{\alpha}\left(1-s_{\jmath}^{p}\right)^{\beta}}, h_{\imath}^{\alpha} h_{\jmath}^{\beta}, d_{\imath}^{\alpha} d_{\jmath}^{\beta}\right)$$

$$\bigotimes_{\imath=1}^{\underline{n}} \bigotimes_{\jmath=\imath}^{\underline{n}} \left(\alpha T_{\imath} \oplus \beta T_{\jmath}\right)$$
$$= \left(\sqrt[p]{\prod_{\imath=1,\jmath=\imath}^{\underline{n}} \left(1-\left(1-s_{\imath}^{p}\right)^{\alpha}\left(1-s_{\jmath}^{p}\right)^{\beta}\right)}, \sqrt[p]{1-\prod_{\imath=1,\jmath=\imath}^{\underline{n}} \left(1-h_{\imath}^{\alpha} h_{\jmath}^{\beta}\right)}, \sqrt[p]{1-\prod_{\imath=1,\jmath=\imath}^{\underline{n}} \left(1-d_{\imath}^{\alpha} d_{\jmath}^{\beta}\right)}\right)$$

$$\left(\bigotimes_{\imath=1}^{\underline{n}} \bigotimes_{\jmath=\imath}^{\underline{n}} \left(\alpha T_{\imath} \oplus \beta T_{\jmath}\right)\right)^{\frac{2}{\underline{n}(\underline{n}+1)}} = \left(\begin{array}{c} \left(\sqrt[p]{\prod_{\imath=1,\jmath=\imath}^{\underline{n}} \left(1-\left(1-s_{\imath}^{p}\right)^{\alpha}\left(1-s_{\jmath}^{p}\right)^{\beta}\right)}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}, \\ \sqrt[p]{1-\prod_{\imath=1,\jmath=\imath}^{\underline{n}} \left(\left(1-h_{\imath}^{\alpha} h_{\jmath}^{\beta}\right)\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}}, \\ \sqrt[p]{1-\prod_{\imath=1,\jmath=\imath}^{\underline{n}} \left(\left(1-d_{\imath}^{\alpha} d_{j}^{\beta}\right)\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}} \end{array}\right)$$

$$\frac{1}{\alpha+\beta}\left(\bigotimes_{\imath=1}^{\underline{n}} \bigotimes_{\jmath=\imath}^{\underline{n}} \left(\alpha T_{\imath} \oplus \beta T_{\jmath}\right)\right)^{\frac{2}{\underline{n}(\underline{n}+1)}} = \left(\begin{array}{c} \left(\sqrt[p]{1-\prod_{\imath=1,\jmath=\imath}^{\underline{n}} \left(1-s_{\imath}^{\alpha p} s_{\jmath}^{\beta p}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}}\right)^{\frac{1}{\alpha+\beta}}, \\ \sqrt[p]{1-\left(1-\prod_{\imath=1,\jmath=\imath}^{\underline{n}} \left(1-\left(1-h_{\imath}^{p}\right)^{\alpha}\left(1-h_{\jmath}^{p}\right)^{\beta}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}\right)^{\frac{1}{\alpha+\beta}}}, \\ \sqrt[p]{1-\left(1-\prod_{\imath=1,\jmath=\imath}^{\underline{n}} \left(1-\left(1-d_{\imath}^{p}\right)^{\alpha}\left(1-d_{\jmath}^{p}\right)^{\beta}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}\right)^{\frac{1}{\alpha+\beta}}} \end{array}\right)$$

Hence,

$$TSFGHM(T_1, T_2, \ldots, T_{\underline{n}}) = \left(\begin{array}{c} \left(\sqrt[p]{1 - \prod_{i=1, j=i}^{\underline{n}} \left(1 - s_i^{\alpha p} s_j^{\beta p}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}} \right)^{\frac{1}{\alpha+\beta}}, \\ \sqrt[p]{1 - \left(1 - \prod_{i=1, j=i}^{\underline{n}} \left(1 - \left(1 - h_i^p\right)^{\alpha} \left(1 - h_j^p\right)^{\beta}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}\right)^{\frac{1}{\alpha+\beta}}}, \\ \sqrt[p]{1 - \left(1 - \prod_{i=1, j=i}^{\underline{n}} \left(1 - \left(1 - d_i^p\right)^{\alpha} \left(1 - d_j^p\right)^{\beta}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}\right)^{\frac{1}{\alpha+\beta}}} \end{array} \right)$$

Hence Proved

We can easily show that TSFGHM operators satisfy the properties given below.

1. **Idempotency:** If all T_i are equal, as $T_i = T$ for all i, then:

$$TSFGHM(T_1, T_2, T_3, \ldots, T_{\underline{n}}) = T \tag{22}$$

2. **Boundedness:** Suppose that T_i be some family of TSFNs, and suppose

$$T^- = \min_i T_i, T^+ = \max_i T_i$$

Then

$$T^- \leq TSFGHM(T_1, T_2, T_3, \ldots, T_{\underline{n}}) \leq T^+ \tag{23}$$

3. **Monotonicity:** suppose T_i and $\acute{T}_i$ be the two families of TSFNs, if $T_i \leq \acute{T}_i$, then

$$TSFGHM(T_1, T_2, T_3, \ldots, T_{\underline{n}}) \leq TSFGHM(\acute{T}_1, \acute{T}_2, \acute{T}_3, \ldots, \acute{T}_{\underline{n}}) \tag{24}$$

By Definition 8, we can see that the TSFGHM operator does not take into account the weights of the information. In many situations, particularly in MADM, the weights of characteristics (attributes) play a key role in the procedure of the aggregation. Therefore, we introduce the TSFWGHM operator as given below.

Definition 9: Let $T_i = (s_i, h_i, d_i)$ be some family of TSFNs, and $\alpha, \beta > 0$. Then TSFWGHM operator is a function $T^{\underline{n}} \to T$ defined as:

$$TSFWGHM(T_1, T_2, T_3, \ldots \ldots \ldots, T_{\underline{n}}) = \frac{1}{\alpha+\beta} \left(\bigotimes_{i=1}^{\underline{n}} \bigotimes_{j=i}^{\underline{n}} \left(\alpha T_i \oplus \beta T_j\right)^{w_i w_j} \right) \tag{25}$$

Based on Equation 25 and Definition 1, we have the following theorem.

Theorem 4: Let $T_i = (s_i, h_i, d_i)$ be some family of TSFNs, and $\alpha, \beta > 0$. Then the aggregated value by using the TSFWGHM operator is a TSFN given by

$$TSFWGHM(T_1, T_2, T_3, \dots\dots, T_{\underline{n}}) = \left(\begin{array}{c} \sqrt[p]{1 - \left(1 - \prod_{i=1, j=i}^{\underline{n}} \left(1 - (1 - s_i^p)^\alpha (1 - s_j^p)^\beta\right)^{w_i w_j}\right)^{\frac{1}{\alpha+\beta}}}, \\ \left(\sqrt[p]{1 - \prod_{i=1, j=i}^{\underline{n}} \left(1 - h_i^{\alpha p} h_j^{\beta p}\right)^{w_i w_j}}\right)^{\frac{1}{\alpha+\beta}}, \\ \left(\sqrt[p]{1 - \prod_{i=1, j=i}^{\underline{n}} \left(1 - d_i^{\alpha p} d_j^{\beta p}\right)^{w_i w_j}}\right)^{\frac{1}{\alpha+\beta}} \end{array} \right) \quad (26)$$

Noticed that the theory in Equation 26 is the generalized form of the weighted geometric Heronian mean operators based on FSs, IFSs, PyFSs, PFSs, and SFSs. Further, the proposed work in Equation 26 is also the modified version of the weighted averaging and weighted geometric aggregation operators for the theory of FSs, IFSs, PyFSs, PFSs, and SFSs.

Proof: By Definition 1.

$$\alpha T_i = \left(\sqrt[p]{1 - (1 - s_i^p)^\alpha}, h_i^\alpha, d_i^\alpha\right), \beta T_j = \left(\sqrt[p]{1 - (1 - s_j^p)^\beta}, h_j^\beta, d_j^\beta\right)$$

Thus,

$$\alpha T_i \oplus \beta T_j = \left(\sqrt[p]{1 - (1 - s_i^p)^\alpha (1 - s_j^p)^\beta}, h_i^\alpha h_j^\beta, d_i^\alpha d_j^\beta\right)$$

Furthermore,

$$(\alpha T_i \oplus \beta T_j)^{w_i w_j}$$

$$= \left(\left(\sqrt[p]{1 - (1 - s_i^p)^\alpha (1 - s_j^p)^\beta}\right)^{w_i w_j}, \left(\sqrt[p]{1 - \left(1 - h_i^{\alpha p} h_j^{\beta p}\right)^{w_i w_j}}\right), \left(\sqrt[p]{1 - \left(1 - d_i^{\alpha p} d_j^{\beta p}\right)^{w_i w_j}}\right)\right)$$

$$\bigotimes_{i=1}^{\underline{n}} \bigotimes_{j=1}^{\underline{n}} (\alpha T_i \oplus \beta T_j)^{w_i w_j} = \left(\begin{array}{c} \left(\sqrt[p]{\prod_{i=1, j=i}^{\underline{n}} \left(1 - (1 - s_i^p)^\alpha (1 - s_j^p)^\beta\right)}\right)^{w_i w_j}, \\ \sqrt[p]{1 - \prod_{i=1, j=i}^{\underline{n}} \left(1 - h_i^{\alpha p} h_j^{\beta p}\right)^{w_i w_j}}, \\ \sqrt[p]{1 - \prod_{i=1, j=i}^{\underline{n}} \left(1 - d_i^{\alpha p} d_j^{\beta p}\right)^{w_i w_j}} \end{array} \right)$$

Now,

$$\frac{1}{\alpha+\beta}\left(\underset{\mathrm{i}=1}{\overset{\mathrm{n}}{\otimes}}\underset{\mathrm{j}=1}{\overset{\mathrm{n}}{\otimes}}\left(\alpha T_{\mathrm{i}}\oplus\beta T_{\mathrm{j}}\right)^{w_{\mathrm{i}}w_{\mathrm{j}}}\right)=\left(\begin{array}{c}\sqrt[p]{1-\left(1-\prod_{\mathrm{i}=1,\mathrm{j}=\mathrm{i}}^{\mathrm{n}}\left(1-\left(1-s_{\mathrm{i}}^{p}\right)^{\alpha}\left(1-s_{\mathrm{j}}^{p}\right)^{\beta}\right)^{w_{\mathrm{i}}w_{\mathrm{j}}}\right)^{\frac{1}{\alpha+\beta}}},\\ \left(\sqrt[p]{1-\prod_{\mathrm{i}=1,\mathrm{j}=\mathrm{i}}^{\mathrm{n}}\left(1-h_{\mathrm{i}}^{\alpha p}h_{\mathrm{j}}^{\beta p}\right)^{w_{\mathrm{i}}w_{\mathrm{j}}}}\right)^{\frac{1}{\alpha+\beta}},\\ \left(\sqrt[p]{1-\prod_{\mathrm{i}=1,\mathrm{j}=\mathrm{i}}^{\mathrm{n}}\left(1-d_{\mathrm{i}}^{\alpha p}d_{\mathrm{j}}^{\beta p}\right)^{w_{\mathrm{i}}w_{\mathrm{j}}}}\right)^{\frac{1}{\alpha+\beta}}\end{array}\right)$$

Hence,

$$TSFWGHM\left(T_1,T_2,T_3,\dots\dots..,T_{\mathrm{n}}\right)=\left(\begin{array}{c}\sqrt[p]{1-\left(1-\prod_{\mathrm{i}=1,\mathrm{j}=\mathrm{i}}^{\mathrm{n}}\left(1-\left(1-s_{\mathrm{i}}^{p}\right)^{\alpha}\left(1-s_{\mathrm{j}}^{p}\right)^{\beta}\right)^{w_{\mathrm{i}}w_{\mathrm{j}}}\right)^{\frac{1}{\alpha+\beta}}},\\ \left(\sqrt[p]{1-\prod_{\mathrm{i}=1,\mathrm{j}=\mathrm{i}}^{\mathrm{n}}\left(1-h_{\mathrm{i}}^{\alpha p}h_{\mathrm{j}}^{\beta p}\right)^{w_{\mathrm{i}}w_{j}}}\right)^{\frac{1}{\alpha+\beta}},\\ \left(\sqrt[p]{1-\prod_{\mathrm{i}=1,\mathrm{j}=\mathrm{i}}^{\mathrm{n}}\left(1-d_{\mathrm{i}}^{\alpha p}d_{\mathrm{j}}^{\beta p}\right)^{w_{\mathrm{i}}w_{\mathrm{j}}}}\right)^{\frac{1}{\alpha+\beta}}\end{array}\right)$$

Proved the theorem.

We can easily show that TSFWGHM operators satisfy the properties given below.

1. **Idempotency:** If all T_{i} are equal, as $T_{\mathrm{i}}=T$ for all i, then:

$$TSFWGHM\left(T_1,\ T_2,T_3,\dots,T_{\mathrm{n}}\right)=T \tag{27}$$

2. **Boundedness:** Suppose that T_{i} be some family of TSFNs, and suppose

$$T^{-}=\min_{\mathrm{i}} T_{\mathrm{i}},T^{+}=\max_{\mathrm{i}} T_{\mathrm{i}}$$

Then

$$T^{-}\leq TSFWGHM\left(T_1,\ T_2,T_3,\dots,T_{\mathrm{n}}\right)\leq T^{+} \tag{28}$$

3. **Monotonicity:** suppose T_{i} and $\acute{T}_{\mathrm{i}}$ be the two families of TSFNs, if $T_{\mathrm{i}}\leq\ \acute{T}_{\mathrm{i}}$, then

$$TSFWGHM\left(T_1,\ T_2,T_3,\dots,T_{\mathrm{n}}\right)\leq TSFWGHM\left(\acute{T}_1,\acute{T}_2,\acute{T}_3,\dots,\acute{T}_{\mathrm{n}}\right) \tag{29}$$

An example to illustrate the applicability of the TSFWGHM operator is given as follows:

Example 2: Let $T_1=(0.4,0.6,0.5)$, $T_2=(0.7,0.5,0.8)$, $T_3=(0.3,0.5,0.3)$ and $T_4=(0.4,0.8,0.4)$ be four TSFNs for $p=3$ and also let $\alpha=\beta=1$ and $w=(0.25,0.35,0.27,0.13)$. Then

$$TSFWGHM(T_1,T_2,T_3,T_4)$$

$$= \left(\begin{array}{c} \sqrt[3]{1-\begin{pmatrix} 1-(1-(1-0.4^3)^1(1-0.4^3)^1)^{0.25^2}(1-(1-0.7^3)^1(1-0.7^3)^1)^{0.35^2} \\ (1-(1-0.3^3)^1(1-0.3^3)^1)^{0.27^2}(1-(1-0.4^3)^1(1-0.4^3)^1)^{0.13^2} \end{pmatrix}^{\frac{1}{2}}}, \\ \left(\sqrt[3]{1-\{(1-0.6^3 0.6^3)^{0.25^2}(1-0.5^3 0.5^3)^{0.35^2}(1-0.5^3 0.5^3)^{0.27^2}(1-0.8^3 0.8^3)^{0.13^2}\}}\right)^{\frac{1}{2}}, \\ \left(\sqrt[3]{1-\{(1-0.5^3 0.5^3)^{0.25^2}(1-0.8^3 0.8^3)^{0.35^2}(1-0.3^3 0.3^3)^{0.27^2}(1-0.4^3 0.4^3)^{0.13^2}\}}\right)^{\frac{1}{2}} \end{array} \right)$$

$$TSFWGHM(T_1,T_2,T_3,T_4) = (0.998043, 0.770866, 0.761736)$$

5 | SOME SPECIAL CASES OF THE GTSFHM OPERATORS AND TSFGHM OPERATORS

This section aims to observe the GTSFHM operator and TSFWGHM operator because of some restrictions. By doing so, we prove the superiority of the GTSFHM operator and TSFEGHM operator over the HM operators of previous fuzzy frameworks.

Consider the GTSFHM operator and TSFWGHM operator as follows:

$$GTSFHM\ (T_1,T_2,\dots,T_{\underline{n}}) = \left(\begin{array}{c} \left(\sqrt[p]{1-\prod_{i=1,j=i}^{\underline{n}}\left(1-s_i^{\alpha p}s_j^{\beta p}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}}\right)^{\frac{1}{\alpha+\beta}}, \\ \sqrt[p]{1-\left(1-\prod_{i=1,j=i}^{\underline{n}}\left(1-\left(1-h_i^p\right)^{\alpha}\left(1-h_j^p\right)^{\beta}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}\right)^{\frac{1}{\alpha+\beta}}}, \\ \sqrt[p]{1-\left(1-\prod_{i=1,j=i}^{\underline{n}}\left(1-\left(1-d_i^p\right)^{\alpha}\left(1-d_j^p\right)^{\beta}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}\right)^{\frac{1}{\alpha+\beta}}} \end{array} \right) \tag{30}$$

$$TSFWGHM(T_1,T_2,T_3,\dots\dots..,T_{\underline{n}}) = \left(\begin{array}{c} \sqrt[p]{1-\left(1-\prod_{i=1,j=i}^{\underline{n}}\left(1-\left(1-s_i^p\right)^{\alpha}\left(1-s_j^p\right)^{\beta}\right)^{w_i w_j}\right)^{\frac{1}{\alpha+\beta}}}, \\ \left(\sqrt[p]{1-\prod_{i=1,j=i}^{\underline{n}}\left(1-h_i^{\alpha p}h_j^{\beta p}\right)^{w_i w_j}}\right)^{\frac{1}{\alpha+\beta}}, \\ \left(\sqrt[p]{1-\prod_{i=1,j=i}^{\underline{n}}\left(1-d_i^{\alpha p}d_j^{\beta p}\right)^{w_i w_j}}\right)^{\frac{1}{\alpha+\beta}} \end{array} \right) \tag{31}$$

If we take $h_i = h_j = 0$. Then GTSFHM and TSFGWHM operators result in the HM operators in the environment of q-ROFSs given as follows.

$$q-ROFGHM\ (T_1,T_2,\dots,T_{\underline{n}}) = \left(\begin{array}{c} \left(\sqrt[p]{1-\prod_{i=1,j=i}^{\underline{n}}\left(1-s_i^{\alpha p}s_j^{\beta p}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}}\right)^{\frac{1}{\alpha+\beta}}, \\ \sqrt[p]{1-\left(1-\prod_{i=1,j=i}^{\underline{n}}\left(1-\left(1-d_i^p\right)^{\alpha}\left(1-d_j^p\right)^{\beta}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}\right)^{\frac{1}{\alpha+\beta}}} \end{array} \right) \tag{32}$$

$$q-ROFWGHM(T_1,T_2,T_3,\ldots\ldots..,T_{\text{n}})=\left(\begin{array}{c}\sqrt[p]{1-\left(1-\prod_{\text{i}=1,\text{j}=\text{i}}^{\text{n}}\left(1-\left(1-s_{\text{i}}^{p}\right)^{\alpha}\left(1-s_{\text{j}}^{p}\right)^{\beta}\right)^{w_{\text{i}}w_{\text{j}}}\right)^{\frac{1}{\alpha+\beta}}},\\ \left(\sqrt[p]{1-\prod_{\text{i}=1,\text{j}=\text{i}}^{\text{n}}\left(1-d_{\text{i}}^{\alpha p}d_{\text{j}}^{\beta p}\right)^{w_{\text{i}}w_{\text{j}}}}\right)^{\frac{1}{\alpha+\beta}}\end{array}\right) \quad (33)$$

If we take $p=2$.Then GTSFHM and TSFGWHM operators result in the HM operators in the environment of SFSs as follows.

$$GSFHM\ (T_1,T_2,\ldots,T_{\text{n}})=\left(\begin{array}{c}\left(\sqrt{1-\prod_{\text{i}=1,\text{j}=\text{i}}^{\text{n}}\left(1-s_{\text{i}}^{2\alpha}s_{\text{j}}^{2\beta}\right)^{\frac{2}{\text{n}(\text{n}+1)}}}\right)^{\frac{1}{\alpha+\beta}},\\ \sqrt{1-\left(1-\prod_{\text{i}=1,\text{j}=\text{i}}^{\text{n}}\left(1-\left(1-h_{\text{i}}^{2}\right)^{\alpha}\left(1-h_{\text{j}}^{2}\right)^{\beta}\right)^{\frac{2}{\text{n}(\text{n}+1)}}\right)^{\frac{1}{\alpha+\beta}}},\\ \sqrt{1-\left(1-\prod_{\text{i}=1,\text{j}=\text{i}}^{\text{n}}\left(1-\left(1-d_{\text{i}}^{2}\right)^{\alpha}\left(1-d_{\text{j}}^{2}\right)^{\beta}\right)^{\frac{2}{\text{n}(\text{n}+1)}}\right)^{\frac{1}{\alpha+\beta}}}\end{array}\right) \quad (34)$$

$$FWGHM(T_1,T_2,T_3,\ldots\ldots..,T_{\text{n}}) \quad (35)$$
$$=\left(\begin{array}{c}\sqrt{1-\left(1-\prod_{\text{i}=1,\text{j}=\text{i}}^{\text{n}}\left(1-\left(1-s_{\text{i}}^{2}\right)^{\alpha}\left(1-s_{\text{j}}^{2}\right)^{\beta}\right)^{w_{\text{i}}w_{\text{j}}}\right)^{\frac{1}{\alpha+\beta}}},\left(\sqrt{1-\prod_{\text{i}=1,\text{j}=\text{i}}^{\text{n}}\left(1-h_{\text{i}}^{2\alpha}h_{\text{j}}^{2\beta}\right)^{w_{\text{i}}w_{\text{j}}}}\right)^{\frac{1}{\alpha+\beta}},\\ \left(\sqrt{1-\prod_{\text{i}=1,\text{j}=\text{i}}^{\text{n}}\left(1-d_{\text{i}}^{2\alpha}d_{\text{j}}^{2\beta}\right)^{w_{\text{i}}w_{\text{j}}}}\right)^{\frac{1}{\alpha+\beta}}\end{array}\right)$$

1) If we take $p=2$ and $h_{\text{i}}=h_{\text{j}}=0$. Then GTSFHM and TSFGWHM operators result in the HM operators in the environment of PyFSs as follows.

$$GPYFHM\ (T_1,T_2,\ldots,T_{\text{n}})=\left(\begin{array}{c}\left(\sqrt{1-\prod_{\text{i}=1,\text{j}=\text{i}}^{\text{n}}\left(1-s_{\text{i}}^{2\alpha}s_{\text{j}}^{2\beta}\right)^{\frac{2}{\text{n}(\text{n}+1)}}}\right)^{\frac{1}{\alpha+\beta}},\\ \sqrt{1-\left(1-\prod_{\text{i}=1,\text{j}=\text{i}}^{\text{n}}\left(1-\left(1-d_{\text{i}}^{2}\right)^{\alpha}\left(1-d_{\text{j}}^{2}\right)^{\beta}\right)^{\frac{2}{\text{n}(\text{n}+1)}}\right)^{\frac{1}{\alpha+\beta}}}\end{array}\right) \quad (36)$$

$$PYFWGHM(T_1,T_2,T_3,\ldots\ldots..,T_{\text{n}})=\left(\begin{array}{c}\sqrt{1-\left(1-\prod_{\text{i}=1,\text{j}=\text{i}}^{\text{n}}\left(1-\left(1-s_{\text{i}}^{2}\right)^{\alpha}\left(1-s_{\text{j}}^{2}\right)^{\beta}\right)^{w_{\text{i}}w_{\text{j}}}\right)^{\frac{1}{\alpha+\beta}}},\\ \left(\sqrt{1-\prod_{\text{i}=1,\text{j}=\text{i}}^{\text{n}}\left(1-d_{\text{i}}^{2\alpha}d_{\text{j}}^{2\beta}\right)^{w_{\text{i}}w_{j}}}\right)^{\frac{1}{\alpha+\beta}}\end{array}\right) \quad (37)$$

If we take $p=1$. Then GTSFHM and TSFGWHM operators result in the HM operators in the environment of PFSs as follows.

$$GPFHM\ (T_1, T_2, \dots, T_{\underline{n}}) = \begin{pmatrix} \left(\sqrt{1 - \prod_{i=1,j=i}^{\underline{n}} \left(1 - s_i^{\alpha} s_j^{\beta}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}}\right)^{\frac{1}{\alpha+\beta}}, \\ \sqrt{1 - \left(1 - \prod_{i=1,j=i}^{\underline{n}} \left(1 - (1 - h_i)^{\alpha}\left(1 - h_j\right)^{\beta}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}\right)^{\frac{1}{\alpha+\beta}}}, \\ \sqrt{1 - \left(1 - \prod_{i=1,j=i}^{\underline{n}} \left(1 - (1 - d_i)^{\alpha}\left(1 - d_j\right)^{\beta}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}\right)^{\frac{1}{\alpha+\beta}}} \end{pmatrix} \tag{38}$$

$$PFWGHM\left(T_1, T_2, T_3, \dots \dots .., T_{\underline{n}}\right) = \begin{pmatrix} \sqrt{1 - \left(1 - \prod_{i=1,j=i}^{\underline{n}} \left(1 - (1 - s_i)^{\alpha}\left(1 - s_j\right)^{\beta}\right)^{w_i w_j}\right)^{\frac{1}{\alpha+\beta}}}, \\ \left(\sqrt{1 - \prod_{i=1,j=i}^{\underline{n}} \left(1 - h_i^{\alpha} h_j^{\beta}\right)^{w_i w_j}}\right)^{\frac{1}{\alpha+\beta}}, \\ \left(\sqrt{1 - \prod_{i=1,j=i}^{\underline{n}} \left(1 - d_i^{\alpha} d_j^{\beta}\right)^{w_i w_j}}\right)^{\frac{1}{\alpha+\beta}} \end{pmatrix} \tag{39}$$

2) If we take $p = 1$ and $h_i = h_j = 0$. Then GTSFHM and TSFGWHM operators result in the HM operators in the environment of IFSs as follows.

$$GIFHM\ (T_1, T_2, \dots, T_{\underline{n}}) = \begin{pmatrix} \left(\sqrt{1 - \prod_{i=1,j=i}^{\underline{n}} \left(1 - s_i^{\alpha} s_j^{\beta}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}}\right)^{\frac{1}{\alpha+\beta}}, \\ \sqrt{1 - \left(1 - \prod_{i=1,j=i}^{\underline{n}} \left(1 - (1 - d_i)^{\alpha}\left(1 - d_j\right)^{\beta}\right)^{\frac{2}{\underline{n}(\underline{n}+1)}}\right)^{\frac{1}{\alpha+\beta}}} \end{pmatrix} \tag{40}$$

$$IFWGHM\left(T_1, T_2, T_3, \dots \dots .., T_{\underline{n}}\right) = \begin{pmatrix} \sqrt{1 - \left(1 - \prod_{i=1,j=i}^{\underline{n}} \left(1 - (1 - s_i)^{\alpha}\left(1 - s_j\right)^{\beta}\right)^{w_i w_j}\right)^{\frac{1}{\alpha+\beta}}}, \\ \left(\sqrt{1 - \prod_{i=1,j=i}^{\underline{n}} \left(1 - d_i^{\alpha} d_j^{\beta}\right)^{w_i w_j}}\right)^{\frac{1}{\alpha+\beta}} \end{pmatrix} \tag{41}$$

6 | MULTI ATTRIBUTE DECISION MAKING

By keeping in mind, the GTSFWHM operator and TSFWGHM operator, this section aims to propose a MADM algorithm based on TSF information.

Let $A = \{A_1, A_2, \dots, A_m\}$ be a discrete set of alternatives, $G = \{G_1, G_2, G_3, \dots, G_{\underline{n}}\}$ be the set of attributes, $w = (w_1, w_2, w_3, \dots, w_{\underline{n}})$ describe the weighting vector of the attribute G_i with a property stated earlier. Consider that $T = \left(T_{ij}\right)_{m\times\underline{n}} = \left(s_{ij}, h_{ij}, d_{ij}\right)_{m\times\underline{n}}$ is the TSF decision matrix, where s_{ij} indicates the degree that the alternative A_{i} satisfies the attribute G_{i} given by the decision-maker, $d_{\mathrm{i}\jmath}$ indicates the degree that the alternative A_{i} does not satisfy the attribute G_{i} given by the decision-maker, $s_{\mathrm{i}\jmath} \in [0,1], h_{\mathrm{i}\jmath} \in [0,1], d_{\mathrm{i}\jmath} \in [0,1]$ such that $(s_{\mathrm{i}\jmath})^p + (h_{\mathrm{i}\jmath})^p + (d_{\mathrm{i}\jmath})^p \leq 1, \mathrm{i} = 1,2,3,\ldots,m, \jmath = 1,2,3,\ldots,\underline{\mathrm{n}}$. In the following algorithm, we apply the GTSFWHM operator and TSFWGHM operator in MADM problems with TSF information.

6.1. Algorithm

The MADM algorithm based on the GTSFHM operator and TSFWGHM operator is illustrated as follows:

Step 1: In this step, a decision matrix based on the information of the decision-makers is established using TSFNs where the TD, AD, and FD of the alternatives based on attributes are given. Further, the least value of p is also investigated at this stage.

Step 2: We use the TSFNs provided in the decision matrix and use the following GTSFWHM operator and TSFWGHM operator to aggregate the TSF information.

$$GTSFWHM(T_1, T_2, T_3, \ldots, T_{\underline{\mathrm{n}}}) = \left(\begin{array}{c} \left(\sqrt[p]{1 - \prod_{\mathrm{i}=1,\jmath=\mathrm{i}}^{\underline{\mathrm{n}}} \left(1 - s_{\mathrm{i}}^{\alpha p} s_{\jmath}^{\beta p}\right)^{w_{\mathrm{i}} w_{\jmath}}} \right)^{\frac{1}{\alpha+\beta}}, \\ \sqrt[p]{1 - \left(1 - \prod_{\mathrm{i}=1,\jmath=\mathrm{i}}^{\underline{\mathrm{n}}} \left(1 - (1 - h_{\mathrm{i}}^p)^{\alpha} (1 - h_{\jmath}^p)^{\beta}\right)^{w_{\mathrm{i}} w_{\jmath}}\right)^{\frac{1}{\alpha+\beta}}}, \\ \sqrt[p]{1 - \left(1 - \prod_{\mathrm{i}=1,\jmath=\mathrm{i}}^{\underline{\mathrm{n}}} \left(1 - (1 - d_{\mathrm{i}}^p)^{\alpha} (1 - d_{\jmath}^p)^{\beta}\right)^{w_{\mathrm{i}} w_{\jmath}}\right)^{\frac{1}{\alpha+\beta}}} \end{array} \right)$$

and

$$TSFWGHM(T_1, T_2, T_3, \ldots\ldots.., T_{\underline{\mathrm{n}}}) = \left(\begin{array}{c} \sqrt[p]{1 - \left(1 - \prod_{\mathrm{i}=1,\jmath=\mathrm{i}}^{\underline{\mathrm{n}}} \left(1 - (1 - s_{\mathrm{i}}^p)^{\alpha} (1 - s_{\jmath}^p)^{\beta}\right)^{w_{\mathrm{i}} w_{\jmath}}\right)^{\frac{1}{\alpha+\beta}}}, \\ \left(\sqrt[p]{1 - \prod_{\mathrm{i}=1,\jmath=\mathrm{i}}^{\underline{\mathrm{n}}} \left(1 - h_{\mathrm{i}}^{\alpha p} h_{\jmath}^{\beta p}\right)^{w_{\mathrm{i}} w_{j}}} \right)^{\frac{1}{\alpha+\beta}}, \\ \left(\sqrt[p]{1 - \prod_{\mathrm{i}=1,\jmath=\mathrm{i}}^{\underline{\mathrm{n}}} \left(1 - d_{\mathrm{i}}^{\alpha p} d_{\jmath}^{\beta p}\right)^{w_{\mathrm{i}} w_{\jmath}}} \right)^{\frac{1}{\alpha+\beta}} \end{array} \right)$$

Step 3: We find the score values S of the aggregated information that we obtained in the form of TSFNs in Step 2.

Step 4: Here we rank all the alternatives A_{i} and choose the best from these in the correspondence with the values of scores S.

Step 5: End.

6.2. Numerical Example

In this part of the paper, we apply the algorithm of MADM based on TSF information in a practical problem to see the applicability and usefulness of the proposed GTSFWHM and TSFGHM. Here we adapt the example from [16] to see the applicability of our proposed work.

Consider a company aiming to implement the enterprise resource planning (ERP) system. The selection committee of the company consists of the two senior representatives of IT and the CEO. A search is carried out about the vendors of the ERP systems and after a comprehensive initial search, the selection committee agreed upon four potential EPR systems $A_i(i = 1, 2, 3, 4)$ to be considered as alternatives. Based on the opinion of the experts in ERP systems, four possible attributes are set up to evaluate the four ERP systems. The four attributes include G_1: The functions of the ERP system, G_2: the fitness of the ERP system, G_3: vendor's competence and G_4: the reputation of the vendors. The 4 observed ERP systems are denoted by $A_i(i = 1, 2, 3, 4)$ need evaluation using the algorithm developed in Section 6.1. The weights of the attributes are a key factor, and, in our case, the weight vector is taken as $w = (0.25, 0.35, 0.27, 0.13)$. The selection committee gives their opinion in the form of TSFNs based on the set attributes. The decision matrix based on TSF information about the 4 alternatives is given in Table 1.

Now we apply the MADM algorithm based on the GTSFWHM operator and TSFWGHM operator to the above example to get the optimum ERP system.

Table 1: The T-Spherical fuzzy decision matrix

	G_1			G_2			G_3			G_4		
	s_1	h_1	d_1	s_2	h_2	d_2	s_3	h_3	d_3	s_4	h_4	d_4
A_1	0.4	0.6	0.5	0.7	0.5	0.8	0.3	0.5	0.3	0.4	0.8	0.4
A_2	0.7	0.5	0.4	0.5	0.4	0.4	0.2	0.3	0.8	0.3	0.7	0.5
A_3	0.5	0.3	0.7	0.6	0.7	0.5	0.5	0.8	0.4	0.8	0.4	0.2
A_4	0.8	0.7	0.3	0.3	0.6	0.3	0.6	0.4	0.6	0.7	0.6	0.3

Step1: We examined the value p for which every triplet in Table 1 becomes a TSFN and found that it happened at $p = 3$. All the information in Table 1 is TSF for $p = 3$ and we used $p = 3$ while aggregating the information in Table 1.

Step 2: In this step, first we use the GTSFWHM operator to aggregate the TSF information obtained in Table 1 for $p = 3$ and for various values of α, β as follows in Table 2:

Table 2: The aggregated result using the GTSFWHM operator.

	$\alpha = \beta = 1$	$\alpha = \beta = 2$	$\alpha = \beta = 3$
A_1	(0.499808, 0.998073, 0.999083)	(0.587976, 0.997465, 0.999001)	(0.622955, 0.996596, 0.998982)
A_2	(0.461978, 0.997996, 0.99906)	(0.557458, 0.997445, 0.998995)	(0.600249, 0.996589, 0.99898)
A_3	(0.485137, 0.998195, 0.999013)	(0.581734, 0.997501, 0.998979)	(0.639516, 0.996607, 0.998976)
A_4	(0.538505, 0.998062, 0.998989)	(0.64123, 0.997452, 0.998976)	(0.687274, 0.99659, 0.998975)

	$\alpha=\beta=4$	$\alpha=\beta=10$
A_1	$(0.641365, 0.995655, 0.998977)$	$(0.675928, 0.989698, 0.998975)$
A_2	$(0.623647, 0.995652, 0.998977)$	$(0.668389, 0.989698, 0.998975)$
A_3	$(0.675191, 0.995658, 0.998975)$	$(0.747403, 0.989698, 0.998975)$
A_4	$(0.713142, 0.995652, 0.998975)$	$(0.763874, 0.989698, 0.998975)$

Now we apply the TSFWGHM operator to the TSF information in Table 1 and the aggregated results are portrayed in Table 3.

Table 2: The aggregated result using the TSFWGHM operator.

	$\alpha=\beta=1$	$\alpha=\beta=2$	$\alpha=\beta=3$
A_1	$(0.998043, 0.770866, 0.761736)$	$(0.997452, 0.701539, 0.678895)$	$(0.99659, 0.664352, 0.630269)$
A_2	$(0.998026, 0.698594, 0.741960)$	$(0.997449, 0.622539, 0.663540)$	$(0.996589, 0.581557, 0.619863)$
A_3	$(0.998079, 0.785908, 0.740449)$	$(0.997466, 0.705142, 0.664896)$	$(0.996596, 0.655334, 0.62275)$
A_4	$(0.998157, 0.780989, 0.672639)$	$(0.997491, 0.710468, 0.594784)$	$(0.996604, 0.670561, 0.552601)$
	$\alpha=\beta=4$	$\alpha=\beta=10$	
A_1	$(0.995652, 0.640011, 0.596619)$	$(0.989698, 0.575975, 0.499245)$	
A_2	$(0.995652, 0.554261, 0.590611)$	$(0.989698, 0.477293, 0.511176)$	
A_3	$(0.995655, 0.619345, 0.593864)$	$(0.989698, 0.510511, 0.506645)$	
A_4	$(0.995657, 0.642716, 0.524299)$	$(0.989698, 0.555251, 0.443801)$	

Step 3: This step is based on the scores of the information we obtained in Table 2 and Table 3. To serve the purpose, Definition 2 is used to compute the scores of the said TSF information, and the results are depicted in Table 4 and Table 5 respectively.

Table 2. The Score values of the data obtained using GTSFWHM operators.

	$\alpha=\beta=1$	$\alpha=\beta=2$	$\alpha=\beta=3$	$\alpha=\beta=4$	$\alpha=\beta=10$
A_1	(-0.87240)	(-0.79373)	(-0.75520)	(-0.73311)	(-0.68811)
A_2	(-0.89859)	(-0.82375)	(-0.78067)	(-0.75437)	(-0.69833)
A_3	(-0.88286)	(-0.80007)	(-0.73538)	(-0.68912)	(-0.57942)
A_4	(-0.84081)	(-0.73327)	(-0.67230)	(-0.63424)	(-0.55120)

Table 3: The score values of the data obtained using the TSFWGHM operator.

	$\alpha=\beta=1$	$\alpha=\beta=2$	$\alpha=\beta=3$	$\alpha=\beta=4$	$\alpha=\beta=10$
A_1	(0.552149)	(0.679475)	(0.739438)	(0.774644)	(0.844976)
A_2	(0.585637)	(0.70022)	(0.751633)	(0.780995)	(0.835840)
A_3	(0.588287)	(0.698474)	(0.748310)	(0.777580)	(0.839360)
A_4	(0.690149)	(0.782077)	(0.821101)	(0.842904)	(0.881999)

Step 4: Based on the scores obtained in Table 4 and Table 5, we portrayed the ranking results for various values of α, β in Table 6 and Table 7 respectively as follows.

Table 4: Ranking by using GTSFWHM operators for various values of α, β

Parameter at different values	Ranking
$\alpha = \beta = 1$	$A_4 > A_1 > A_3 > A_2$
$\alpha = \beta = 2$	$A_4 > A_1 > A_3 > A_2$
$\alpha = \beta = 3$	$A_4 > A_3 > A_1 > A_2$
$\alpha = \beta = 4$	$A_4 > A_3 > A_1 > A_2$
$\alpha = \beta = 10$	$A_4 > A_3 > A_1 > A_2$

Table 5: Ranking by using TSFWGHM operators for various values of α, β

Parameter at different values	Ranking
$\alpha = \beta = 1$	$A_4 > A_3 > A_2 > A_1$
$\alpha = \beta = 2$	$A_4 > A_2 > A_3 > A_1$
$\alpha = \beta = 3$	$A_4 > A_2 > A_3 > A_1$
$\alpha = \beta = 4$	$A_4 > A_2 > A_3 > A_1$
$\alpha = \beta = 10$	$A_4 > A_1 > A_3 > A_2$

From the ranking results obtained in Table 6 and Table 7, alternative A_4 comes out to be the best one for all the values of α, β. However, some changes must occur in the ranking pattern of the alternatives A_1, A_2 and A_3. We also observed that the ranking pattern only changes for $\alpha = \beta = 10$ and no change occurs after this value. All this demonstrates a clear picture of the variation in the ranking results for different values of α, β. For further demonstration, the results of Table 6 and Table 7 are depicted in Figures 1-5 respectively.

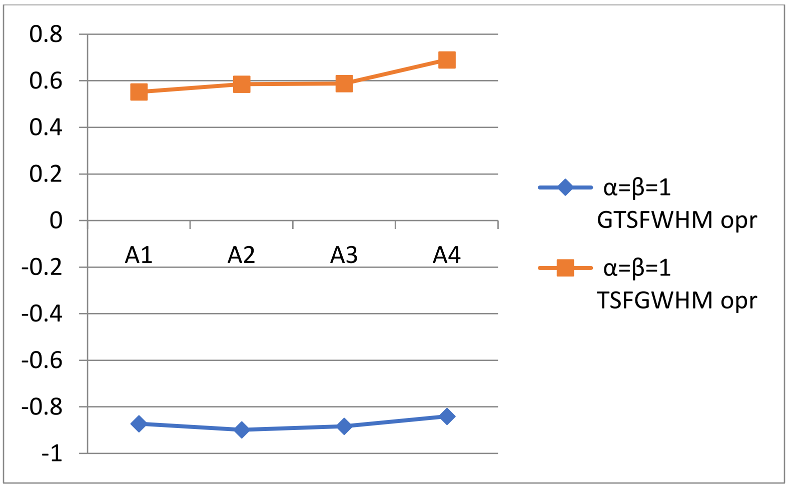

Figure 1: (Ranking Results for $\alpha = \beta = 1$)

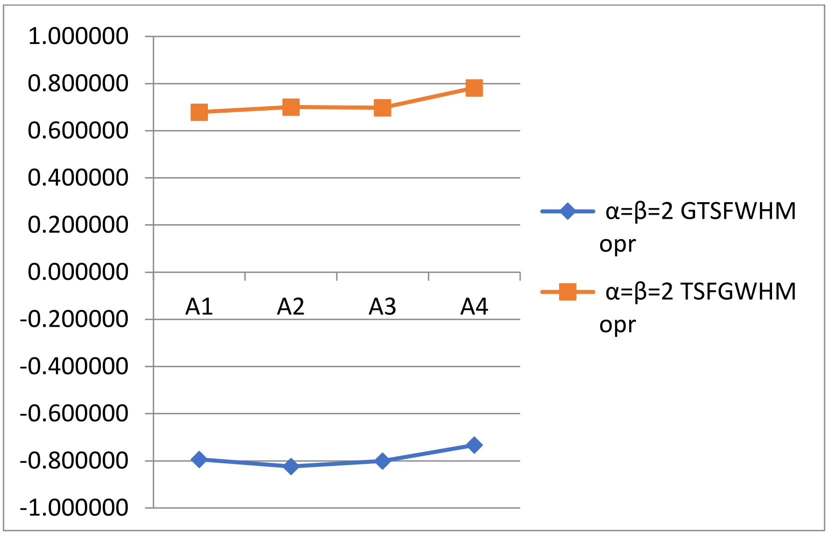

Figure 2: (Ranking Results for α=β=2)

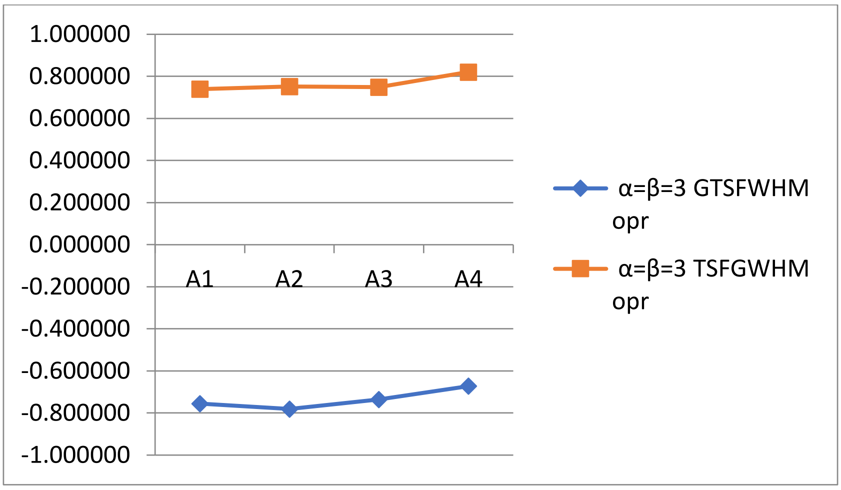

Figure 3: (Ranking Results for α=β=3)

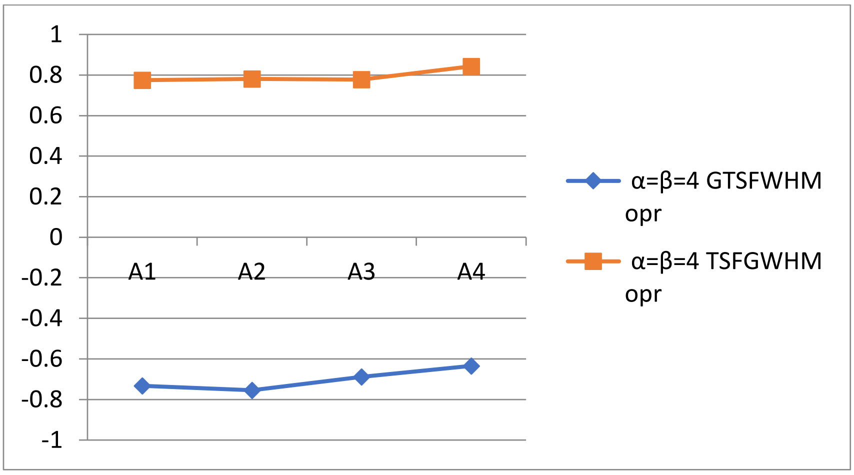

Figure 4: (Ranking Results for α=β=4)

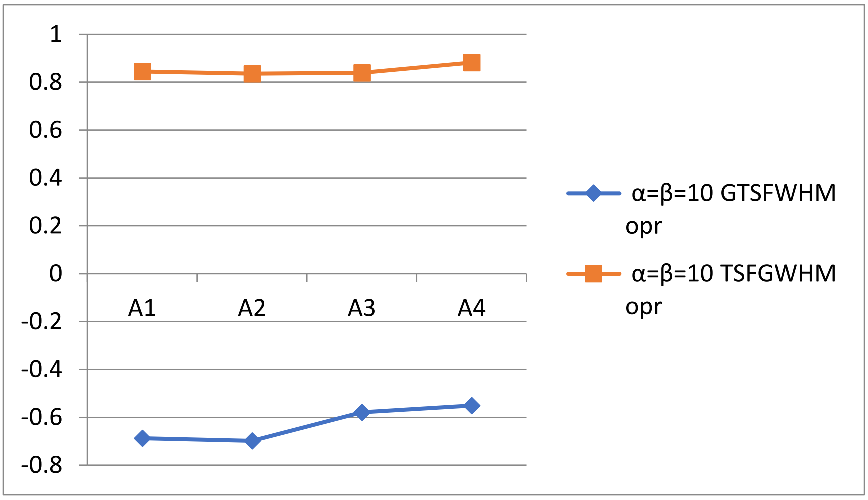

Figure 5: (Ranking Results for α=β=10)

Here we observed that parameters α and β had a critical role in the results. Now we observe the influence of the parameter p on the ranking results and a brief demonstration of the ranking results using the GTSFWHM operator and TSFWGHM operator are displayed in Table 8 and Table 9 respectively.

Table 6: The score values using the GTSFWHM operator for various values of *p*

	$p = 3$	$p = 4$	$p = 5$	$p = 6$	$p = 7$
A_1	(-0.87240)	(-0.91166)	(-0.93778)	(-0.95568)	(-0.96811)
A_2	(-0.89859)	(-0.93240)	(-0.95333)	(-0.96698)	(-0.97616)
A_3	(-0.88286)	(-0.92153)	(-0.94412)	(-0.95818)	(-0.96754)
A_4	(-0.84081)	(-0.88019)	(-0.90752)	(-0.92741)	(-0.94235)

Table 7: The score values using the TSFWGHM operator for various values of *p*

	$p = 3$	$p = 4$	$p = 5$	$p = 6$	$p = 7$
A_1	(0.552149)	(0.638016)	(0.704201)	(0.75637)	(0.798125)
A_2	(0.585637)	(0.673367)	(0.739367)	(0.790159)	(0.829874)
A_3	(0.588287)	(0.676622)	(0.742989)	(0.793922)	(0.833602)
A_4	(0.690149)	(0.773961)	(0.832564)	(0.874533)	(0.905066)

Table 8 and Table 9 show the ranking of the aggregated data using the GTSFWHM operator and TSFWGHM operator. It can be seen that upon changing the parameter p to higher values, there is no significant change in the ranking of the alternatives. In the case of Ullah et al [49], the ranking results drastically got changed by increasing the value of p. All this shows that the change in the ranking results by varying p does not occur always and can be seen from Table 10 and Table 11.

Table 8: Ranking by using GTSFWHM operators for various values of *p*

Parameter at different values	Ranking
$p = 3$	$A_4 > A_1 > A_3 > A_2$
$p = 4$	$A_4 > A_1 > A_3 > A_2$
$p = 5$	$A_4 > A_1 > A_3 > A_2$
$p = 6$	$A_4 > A_1 > A_3 > A_2$
$p = 7$	$A_4 > A_1 > A_3 > A_2$

Table 9: Ranking by using TSFWGHM operators for various values of p

Parameter at different values	Ranking
$\mathbf{p = 3}$	$\mathrm{A}_4 > \mathrm{A}_3 > \mathrm{A}_2 > \mathrm{A}_1$
$\mathbf{p = 4}$	$\mathrm{A}_4 > \mathrm{A}_3 > \mathrm{A}_2 > \mathrm{A}_1$
$\mathbf{p = 5}$	$\mathrm{A}_4 > \mathrm{A}_3 > \mathrm{A}_2 > \mathrm{A}_1$
$\mathbf{p = 6}$	$\mathrm{A}_4 > \mathrm{A}_3 > \mathrm{A}_2 > \mathrm{A}_1$
$\mathbf{p = 7}$	$\mathrm{A}_4 > \mathrm{A}_3 > \mathrm{A}_2 > \mathrm{A}_1$

7 | COMPARATIVE ANALYSIS

The prominent characteristic of the GTSFWHM and TSFWGHM operators is that these operators handle information based on human opinion with less information loss. These operators not only handle the TSF information but can also be applied to the information described in other fuzzy frameworks such as IFS, PyFS, q-ROFS, PFS, SFS, etc. We utilized the existing TSF aggregation operators in the decision matrix obtained in Table 1 and listed the results in Table 12 below. Further, it is important to note that the previously existing HMOs cannot be applied to information discussed in this paper in the TSF environment.

Table 10: Results by applying different AOs on TSF information.

Aggregation Operators	By	Ranking
TSFHWA	Ullah et al. [49]	$A_4 > A_3 > A_1 > A_2$
TSFHWG	Ullah et al. [49]	$A_4 > A_3 > A_1 > A_2$
TSFWA	Ullah et al. [28]	$A_4 > A_3 > A_1 > A_2$
TSFWG	Ullah et al. [28]	$A_3 > A_4 > A_1 > A_2$
HM operators of IFSs	Yu [40]	Failed
HM operators of PyFS	Li [41]	Failed
HM operators of q-ROFSs	Wei [16]	Failed
HM operators of PFSs	Wei [42]	Failed

The results of Table 12 indicate that the HM operators of the IFSs, PyFSs, q-ROFSs, and PFSs have their limitations when we have to deal with information based on human opinion with the flexibility provided by the TSFSs. This shows the superiority of the TSF aggregation information. The key advantages are as follows:

1) GTSFWHM and TSFWGHM operators are more flexible ground than the HMOs of q-ROFSs, PyFSs, IFSs, PFSs, SFSs, etc.
2) GTSFWHM and TSFWGHM operators can be utilized for more complex human-provided information with less information loss by discussing the TD, AD, FD, and RD of the human opinion. IFS, PyFS, and QROFS use only TD and FD to describe information, and the AD and RD are lost.

After a lot of investigation, we noticed that the proposed work is the generalization of the many existing information such as FSs, IFSs, PyFSs, PFSs, and SFSs. But it also possible to further modify the proposed theory by utilizing the type-3 or interval-valued type-3 fuzzy sets [50] is to enhance the quality of the proposed work.

8 | CONCLUSION

In this paper, we comprehensively discussed the design of TSFS and how it handles uncertain information with more flexibility and less information loss. Then we investigated the role of TSF information aggregation in MADM issues based on the generalized HMOs and geometric HMOs. Some key contributions are as follows:

1) The generalized HMO and geometric HMO are developed in TSF settings.

To deal with the information where weights of information become essential.

2) We proposed a GTSFHM operator and a GTSFWHM operator.
3) We proposed the TSFGHM operator and TSFWGHM operator.
4) A MADM algorithm is set up based on proposed HM operators.
5) The problem of the selection of the ERP system is carried out based on the developed algorithm.
6) The influence of the variable parameters α, β, and p is observed numerically and conclusions are drawn.
7) The comparison of the TSF HM operators and previously developed HM operators is set up and analyzed for possible advantages.

In near future, we aim to extend our work in interval type-3 fuzzy sets [51], [52] and interval-valued TSF and complex TSF environments [53], and then we focus to utilize it in the field of artificial intelligence, machine learning, game theory, and clustering analysis is to enhance the quality of the invented theory.

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