

Schweizer-Sklar Aggregation Operators with Unknown Weight for Picture Fuzzy Information

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ABSTRACT

The Interval-Valued Picture fuzzy set (IV-PFS) possesses multiple advantages in various fields of research and other systematic approaches which provides an innovative method for solving multi-attribute decision-making (MADM) issues. The main purpose of this article is to define some power average aggregation operators (PA AOs) which are helpful to ameliorate some of the adverse consequences of falsified information, and the Schweizer-Sklar (SS) operational rules are used to introduce some novel AOs such as the IV-PFSSPA and IV-PFSSWPA operators. Furthermore, the distinctive characteristics of these provided AOs are discussed in detail. Moreover, we implement some of the suggested AOs to formulate a process for resolving MADM problems concerning IV-PF data. We provide an example to further explain the importance and usefulness of the tree selection scenario which is helpful to protect the environment and also enhance human lives. To demonstrate the distinctive nature and feasibility of the suggested strategy, an analysis of the comparability is carried out with the previously discussed approaches.

Keywords: IV-PF power average operator, IV-PF power weighted average operator, tree plantation scenario, MADM difficulties.

1 | INTRODUCTION

Decision-making (DM) is a comprehensively successful approach for interpreting complex and unreliable data in practical situations. Multi-attribute decision-making (MADM) mechanism has triggered a lot of interest from many different researchers and has been extensively investigated and successfully used in many fields of research [1]–[7]. The MADM technique is one of the authentic and helpful processes that take place in daily life, which is used to rank the limited set of DM concerns because they are so logical and intuitive to the decision-maker. Moreover, the concept of fuzzy set (FS) was initially investigated by Zadeh [8] in 1965, which adopts membership value (MV) $\hat{u}_E$ to characterize unreliable and unpredictable information under fuzzy frameworks. In 1986, Attanasov [9] gave the idea of intuitionistic FS (IFS) which adopts MV $\hat{u}_E$ and non-membership value (NMV) $\hat{v}_E$ to characterize the complex and unpredictable information with the condition $0 \leq (\hat{u}_E + \hat{v}_E) \leq 1$. The AOs of IFSs have been extensively investigated and used in numerous fields of research [10–16]. Based on the generalized version of IFS, the interval-valued

IFS (IV-IFS) was initially investigated by Attanasov and Gargov [17] in 1989 to deal with the ambiguous information by using lower and upper terms of the MV and NMV that satisfied such condition $0 \leq (\hat{u}_E^u) + (\tilde{v}_E^u) \leq 1$. In some cases, the IFS and the IV-IFS are unable to precisely describe ambiguous and inconsistent information. To handle these situations, Yager [18] investigated the idea of Pythagorean FS (PyFS) which adopts MV $\hat{u}_E$, abstinence value (AV) $\tilde{a}_E$, and NMV $\tilde{v}_E$ and their sum of squares lies on the interval $[0, 1]$. A lot of progress has been achieved by applying IFSs and PyFSs for the solution of challenging MADM problems [19-22]. Based on the generalized form of PyFS, Rehman et al. [23] gave the idea of interval-valued PyFS (IV-PyFS) in which the MV $\hat{u}_E$ and NMV must satisfy such limitations $0 \leq (\hat{u}_E^u)^2 + (\tilde{v}_E^u)^2 \leq 1$. Recently, Yager [24] further extended the novel concept of PyFS and proposed the specific description of q-Rung ortho-pair FS (q-ROFS) in which the MV $\hat{u}_E$, AV $\tilde{a}_E$, and NMV $\tilde{v}_E$ must satisfy the condition $((\hat{u}_E^u)^q + (\tilde{v}_E^u)^q) \leq 1, q \in \mathbb{Z}^+$. Definitely the q-ROFS allows higher level of uncertainty than the PyFS and IFS and its scope of application is broader than that of the PFS and IFS. Cuong [25] emphasized the concept of picture FS (PFS) and explored some of the fundamental operations and features of PFS. The PF framework is significant because it may explain the data through four different scenarios using MV $\hat{u}_E$, AV $\tilde{a}_E$, NMV $\tilde{a}_E$, and RV and their sum lies on the interval $[0, 1]$. The AOs of PFSs have been extensively investigated and used in numerous fields of research for solving MADM issues [26-29]. Based on the further extension of PFS, the interval-valued PFS (IV-PFS) was also introduced by Cuong [25] in which the lower and upper terms of MV $\hat{u}_E$, AV $\tilde{a}_E$, and NMV $\tilde{v}_E$ must satisfy such condition $0 \leq ((\hat{u}_E^u) + (\tilde{a}_E^u) + (\tilde{v}_E^u)) \leq 1$. The applications of IV-PFSs [30-31] are broader than that of IFSs, IV-IFSs PFSs, IV-PFSs, q-ROFSs, IV-qROFSs, and PFSs and are used in numerous fields of research for solving MADM issues.

After the development of IV-PFSs, various researchers began started working on different AOs by using IV-PF data to deal with MADM challenges. Jabeen et al. [32] proposed Aczel-Alsina (AA) AOs for IV-PFVs and apply them to MADM issues. Kamaci et al. [33] worked on dynamic AOs (DAOs) and Einstein operators for IVP Hesitant FVs (IV-PFHFVs) to deal with the MADM problems. Mahmood et al. [34-35] modified the idea of AA AOs and generalized similarity measures to diagnose the theory of bipolar complex fuzzy sets (BCFSs) and their application in decision-making. The PAOs play an important role to aggregate fuzzy information. Moreover, the concept of PAOs was initially investigated by Yager [36] in 2001, which is helpful to ameliorate some of the adverse consequences of falsified information provided by the decision-maker. Zhou et al. [37] developed generalized PAOs (GPAOs) for handling MADM issues. Zindani et al. [38] worked on SSPAOs by using the IV-IF framework and established their applications to MADM difficulties.

Various researchers worked on triangular norms by using different operational rules such as AA t-norm (AA-TN), AA t-conorm (AA-TCN), Dombi TN (D-TN), D-TCN, Hamacher TN (H-TN), H-TCN, Einstein TN (E-TN), E-TCN, Algebraic TN (A-TN), and A-TCN and so on. These above-mentioned

triangular norms are used for aggregating statistical information and applied to solve various MADM issues by using different fuzzy scenarios. The specified form of the triangular norm that satisfied the capability requirements is named Schweizer-sklar TN (SS-TN) and SS-TCN. Since the SS-TN and TCN incorporate the parameters, which have the potential to give the information fusion procedure offers greater adaptability and suitability and make them more effectively optimal for handling real-world DM compared to various triangular norms. The most significant and effective behavior of our article is to suggest Schweizer-Sklar operational rules and power aggregation operators (PAOs) based on the IV-PFSs. The PAOs are helpful to ameliorate some of the adverse consequences of falsified information given by the decision-maker (DM). To formulate some novel AOs such as interval-valued picture Schweizer-Sklar power average operators and their weighted form. We also discussed the distinctive characteristics of these newly suggested AOs and use them for MADM challenges.

The rest of this article can be categorized as follows: In section 2 we provided some fundamental concepts as well as findings that will be implemented in our upcoming sections. In section 3, we developed various AOs including IV-PFSSPA, and IV-PFSSPPWA operators, and examined the distinctive characteristics of these provided AOs. In section 4, we applied our proposed AOs to determine the process of solving MADM and provide a numerical description of the problem. In section 5, we discussed the ranking results of our proposed AOs with previously existing AOs. In section 6, we discussed concluded remarks.

2 | PRELIMINARIES

This section consists of some basic definitions of PFS, IV-PFS, distance measuring formula, score function, and PA operator.

Definition 1: [39] A PFS $\mathbb{E}$ on a set $\mathfrak{N}$ is symbolically represented as

$$\mathbb{E} = \{(\mathfrak{t}, \acute{u}_{\mathbb{E}}(\mathfrak{t}), \bar{a}_{\mathbb{E}}(\mathfrak{t}), \tilde{v}_{\mathbb{E}}(\mathfrak{t})) | \mathfrak{t} \in \mathfrak{N}\}, \quad (1)$$

Where $\acute{u}_{\mathbb{E}} : \mathfrak{N} \rightarrow [0, 1]$ is a function that represents the positive value (PV), $\bar{a}_{\mathbb{E}} : \mathfrak{N} \rightarrow [0, 1]$ is a function that represents the abstinence value (AV), and $\tilde{v}_{\mathbb{E}} : \mathfrak{N} \rightarrow [0, 1]$ is a function that represents the negative value (NV) of an element $\mathfrak{t} \in \mathfrak{N}$ to $\mathbb{E}$ respectively, $\forall \mathfrak{t} \in \mathfrak{N}$, with the restriction

$$(\acute{u}_{\mathbb{E}}(\mathfrak{t})) + (\bar{a}_{\mathbb{E}}(\mathfrak{t})) + (\tilde{v}_{\mathbb{E}}(\mathfrak{t})) \leq 1$$

And $\tilde{\eta}_{\mathbb{E}}(\mathfrak{t}) = \left(1 - (\acute{u}_{\mathbb{E}}(\mathfrak{t}) + \bar{a}_{\mathbb{E}}(\mathfrak{t}) + \tilde{v}_{\mathbb{E}}(\mathfrak{t}))\right)$ is a function that represents the refusal value (RV) of an element $\mathfrak{t} \in \mathfrak{N}$ to $\mathbb{E}$. Conveniently, the simple form of PFVs is described as $\mathbb{E} = (\acute{u}_{\mathbb{E}}, \bar{a}_{\mathbb{E}}, \tilde{v}_{\mathbb{E}})$.

Definition 2: [40] An IV-PFS $\mathbb{E}$ on a set $\mathfrak{N}$ is symbolically represented as

$$\mathbb{E} = \{(\mathfrak{t}, [\acute{u}_{\mathbb{E}}^l(\mathfrak{t}), \acute{u}_{\mathbb{E}}^u(\mathfrak{t})], [\bar{a}_{\mathbb{E}}^l(\mathfrak{t}), \bar{a}_{\mathbb{E}}^u(\mathfrak{t})], [\tilde{v}_{\mathbb{E}}^l(\mathfrak{t}), \tilde{v}_{\mathbb{E}}^u(\mathfrak{t})]) | \mathfrak{t} \in \mathfrak{N}\}, \quad (2)$$

Where $0 \leq \acute{u}_{\mathbb{E}}^l(\mathfrak{t}) \leq \acute{u}_{\mathbb{E}}^u(\mathfrak{t}) \leq 1$, $\bar{a}_{\mathbb{E}}^l(\mathfrak{t}) \leq \bar{a}_{\mathbb{E}}^u(\mathfrak{t}) \leq 1$, $\tilde{v}_{\mathbb{E}}^l(\mathfrak{t}) \leq \tilde{v}_{\mathbb{E}}^u(\mathfrak{t}) \leq 1$ and $(\acute{u}_{\mathbb{E}}^u(\mathfrak{t})) + (\bar{a}_{\mathbb{E}}^u(\mathfrak{t})) + (\tilde{v}_{\mathbb{E}}^u(\mathfrak{t})) \leq 1$.

And $\tilde{\eta}_{\mathbb{E}}(\mathfrak{h}) = \left[\left(1 - \left(\dot{u}_{\mathbb{E}}^l(\mathfrak{h}) + \bar{a}_{\mathbb{E}}^l(\mathfrak{h}) + \tilde{v}_{\mathbb{E}}^l(\mathfrak{h}) \right) \right), \left(1 - \left(\dot{u}_{\mathbb{E}}^u(\mathfrak{h}) + \bar{a}_{\mathbb{E}}^u(\mathfrak{h}) + \tilde{v}_{\mathbb{E}}^u(\mathfrak{h}) \right) \right) \right]$ is a function that represents the refusal value (RV) of an element $\mathfrak{h} \in \mathfrak{K}$ to $\mathbb{E}$. Conveniently, the simple form of IV-PFVs is described as $\mathbb{E} = ([\dot{u}_{\mathbb{E}}^l, \dot{u}_{\mathbb{E}}^u], [\bar{a}_{\mathbb{E}}^l, \bar{a}_{\mathbb{E}}^u], [\tilde{v}_{\mathbb{E}}^l, \tilde{v}_{\mathbb{E}}^u])$.

Definition 3: [41] Assume that $\mathbb{E} = ([\dot{u}_{\mathbb{E}}^l, \dot{u}_{\mathbb{E}}^u], [\bar{a}_{\mathbb{E}}^l, \bar{a}_{\mathbb{E}}^u], [\tilde{v}_{\mathbb{E}}^l, \tilde{v}_{\mathbb{E}}^u])$, $\mathbb{E}_1 = ([\dot{u}_{\mathbb{E}_1}^l, \dot{u}_{\mathbb{E}_1}^u], [\bar{a}_{\mathbb{E}_1}^l, \bar{a}_{\mathbb{E}_1}^u], [\tilde{v}_{\mathbb{E}_1}^l, \tilde{v}_{\mathbb{E}_1}^u])$ and $\mathbb{E}_2 = ([\dot{u}_{\mathbb{E}_2}^l, \dot{u}_{\mathbb{E}_2}^u], [\bar{a}_{\mathbb{E}_2}^l, \bar{a}_{\mathbb{E}_2}^u], [\tilde{v}_{\mathbb{E}_2}^l, \tilde{v}_{\mathbb{E}_2}^u])$ be three IV-PFVs. Then

- 1) $\mathbb{E}_1 \subseteq \mathbb{E}_2$ iff $\dot{u}_{\mathbb{E}_1}^l \leq \dot{u}_{\mathbb{E}_2}^l, \dot{u}_{\mathbb{E}_1}^u \leq \dot{u}_{\mathbb{E}_2}^u, \bar{a}_{\mathbb{E}_1}^l \geq \bar{a}_{\mathbb{E}_2}^l, \bar{a}_{\mathbb{E}_1}^u \geq \bar{a}_{\mathbb{E}_2}^u, \tilde{v}_{\mathbb{E}_1}^l \geq \tilde{v}_{\mathbb{E}_2}^l$, and $\tilde{v}_{\mathbb{E}_1}^u \geq \tilde{v}_{\mathbb{E}_2}^u$.
- 2) $\mathbb{E}_1 = \mathbb{E}_2$ iff $\mathbb{E}_1 \subseteq \mathbb{E}_2$ and $\mathbb{E}_2 \subseteq \mathbb{E}_1$
- 3) $\mathbb{E}_1 \cap \mathbb{E}_2 = ([\min\{\dot{u}_{\mathbb{E}_1}^l, \dot{u}_{\mathbb{E}_2}^l\}, \min\{\dot{u}_{\mathbb{E}_1}^u, \dot{u}_{\mathbb{E}_2}^u\}], [\max\{\bar{a}_{\mathbb{E}_1}^l, \bar{a}_{\mathbb{E}_2}^l\}, \max\{\bar{a}_{\mathbb{E}_1}^u, \bar{a}_{\mathbb{E}_2}^u\}], [\max\{\tilde{v}_{\mathbb{E}_1}^l, \tilde{v}_{\mathbb{E}_2}^l\}, \max\{\tilde{v}_{\mathbb{E}_1}^u, \tilde{v}_{\mathbb{E}_2}^u\}])$
- 4) $\mathbb{E}_1 \cup \mathbb{E}_2 = ([\max\{\dot{u}_{\mathbb{E}_1}^l, \dot{u}_{\mathbb{E}_2}^l\}, \max\{\dot{u}_{\mathbb{E}_1}^u, \dot{u}_{\mathbb{E}_2}^u\}], [\min\{\bar{a}_{\mathbb{E}_1}^l, \bar{a}_{\mathbb{E}_2}^l\}, \min\{\bar{a}_{\mathbb{E}_1}^u, \bar{a}_{\mathbb{E}_2}^u\}], [\min\{\tilde{v}_{\mathbb{E}_1}^l, \tilde{v}_{\mathbb{E}_2}^l\}, \min\{\tilde{v}_{\mathbb{E}_1}^u, \tilde{v}_{\mathbb{E}_2}^u\}])$
- 5) $\mathbb{E}^c = ([\tilde{v}_{\mathbb{E}}^l, \tilde{v}_{\mathbb{E}}^u], [\bar{a}_{\mathbb{E}}^l, \bar{a}_{\mathbb{E}}^u], [\dot{u}_{\mathbb{E}}^l, \dot{u}_{\mathbb{E}}^u])$

Definition 4: [41] Assume that $\mathbb{E} = ([\dot{u}_{\mathbb{E}}^l, \dot{u}_{\mathbb{E}}^u], [\bar{a}_{\mathbb{E}}^l, \bar{a}_{\mathbb{E}}^u], [\tilde{v}_{\mathbb{E}}^l, \tilde{v}_{\mathbb{E}}^u])$ be the IV-PFVs. Then the score function $\mathcal{S}(\mathbb{G})$ is

$$\delta(\mathbb{E}) = \frac{\dot{u}_{\mathbb{E}}^l(1 - \bar{a}_{\mathbb{E}}^l - \tilde{v}_{\mathbb{E}}^l) + \dot{u}_{\mathbb{E}}^u(1 - \bar{a}_{\mathbb{E}}^u - \tilde{v}_{\mathbb{E}}^u)}{3}, \delta(\mathbb{E}) \in [0, 1] \quad (3)$$

Definition 5: [36] Assume that $\mathbb{E}_{\mathfrak{r}}$ ($\mathfrak{r} = 1, 2, \dots, \mathfrak{q}$) is a list of non-negative numbers. Then PA operator of $\mathbb{E}_{\mathfrak{r}}$ is

$$PA(\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3 \dots \mathbb{E}_{\mathfrak{q}}) = \bigoplus_{\mathfrak{r}=1}^{\mathfrak{q}} \frac{(1 + T(\mathbb{E}_{\mathfrak{r}}))}{\sum_{\mathfrak{r}=1}^{\mathfrak{q}} (1 + T(\mathbb{E}_{\mathfrak{r}}))} \mathbb{E}_{\mathfrak{r}} \quad (4)$$

Where $T(\mathbb{E}_{\mathfrak{r}}) = \sum_{\mathfrak{r}=1}^{\mathfrak{q}} s(\mathbb{E}_{\mathfrak{r}}, \mathbb{E}_{\mathfrak{k}})$ and $\dot{S}(\mathbb{E}_{\mathfrak{r}}, \mathbb{E}_{\mathfrak{q}})$ showing the support value (SV) of $\mathbb{E}_{\mathfrak{r}}$ and $\mathbb{E}_{\mathfrak{q}}$.

Definition 6: [42] Schweizer- Sklar ($T\tilde{N}$) and ($T\text{-}\zeta\tilde{N}$) are expressed by

$$\mathcal{Z}(\dot{u}, \bar{a}, \tilde{v}) = (\dot{u}^{\mathfrak{Q}} + \bar{a}^{\mathfrak{Q}} + \tilde{v}^{\mathfrak{Q}} - 1)^{1/\mathfrak{Q}} \quad (5)$$

$$\mathfrak{Q}(\dot{u}, \bar{a}, \tilde{v}) = \left(1 - (1 - (\dot{u}^{\mathfrak{Q}} + \bar{a}^{\mathfrak{Q}} + \tilde{v}^{\mathfrak{Q}} - 1))^{1/\mathfrak{Q}} \right)^{\mathfrak{Q}}$$

Where $\dot{u}, \bar{a}, \tilde{v} \in [0, 1]$ and $\mathfrak{Q} < 0$.

3 | IV-PFPA AOs BY APPLYING SS OPERATIONAL LAWS

This section provides a novel idea for IV-PFPAOs by using SS operational rules including IV-PFSSPA operator and IV-PFSSPWA operators. We also examine the distinctive characteristics of the IVPFSSPA operator and IV-PFSSWPA operator respectively.

Definition 7: Assume that $\mathbb{E}_1 = ([\dot{u}_{\mathbb{E}_1}^l, \dot{u}_{\mathbb{E}_1}^u], [\bar{a}_{\mathbb{E}_1}^l, \bar{a}_{\mathbb{E}_1}^u], [\tilde{v}_{\mathbb{E}_1}^l, \tilde{v}_{\mathbb{E}_1}^u])$ and $\mathbb{E}_2 = ([\dot{u}_{\mathbb{E}_2}^l, \dot{u}_{\mathbb{E}_2}^u], [\bar{a}_{\mathbb{E}_2}^l, \bar{a}_{\mathbb{E}_2}^u], [\tilde{v}_{\mathbb{E}_2}^l, \tilde{v}_{\mathbb{E}_2}^u])$ are two IV-PFVs, then the distance between $[\dot{u}_{\mathbb{E}}^l, \dot{u}_{\mathbb{E}}^u]$, $[\bar{a}_{\mathbb{E}}^l, \bar{a}_{\mathbb{E}}^u]$, and $[\tilde{v}_{\mathbb{E}}^l, \tilde{v}_{\mathbb{E}}^u]$ are expressed by

$$\mathcal{D}(\mathbb{E}_1, \mathbb{E}_2) = \frac{1}{6} \left(|\acute{u}_{\mathbb{E}_1}^l - \acute{u}_{\mathbb{E}_2}^l| + |\acute{u}_{\mathbb{E}_1}^u - \acute{u}_{\mathbb{E}_2}^u| + |\bar{a}_{\mathbb{E}_1}^l - \bar{a}_{\mathbb{E}_2}^l| + |\bar{a}_{\mathbb{E}_1}^u - \bar{a}_{\mathbb{E}_2}^u| + |\tilde{v}_{\mathbb{E}_1}^l - \tilde{v}_{\mathbb{E}_2}^l| + |\tilde{v}_{\mathbb{E}_1}^u - \tilde{v}_{\mathbb{E}_2}^u| \right) \quad (6)$$

Definition 8: Assume that $\mathbb{E} = ([\acute{u}_{\mathbb{E}}^l, \acute{u}_{\mathbb{E}}^u], [\bar{a}_{\mathbb{E}}^l, \bar{a}_{\mathbb{E}}^u], [\tilde{v}_{\mathbb{E}}^l, \tilde{v}_{\mathbb{E}}^u])$, $\mathbb{E}_1 = ([\acute{u}_{\mathbb{E}_1}^l, \acute{u}_{\mathbb{E}_1}^u], [\bar{a}_{\mathbb{E}_1}^l, \bar{a}_{\mathbb{E}_1}^u], [\tilde{v}_{\mathbb{E}_1}^l, \tilde{v}_{\mathbb{E}_1}^u])$ and $\mathbb{E}_2 = ([\acute{u}_{\mathbb{E}_2}^l, \acute{u}_{\mathbb{E}_2}^u], [\bar{a}_{\mathbb{E}_2}^l, \bar{a}_{\mathbb{E}_2}^u], [\tilde{v}_{\mathbb{E}_2}^l, \tilde{v}_{\mathbb{E}_2}^u])$ be any three IV-PFVs, then

$$\mathbb{E}_1 \oplus \mathbb{E}_2 = \left(\left[1 - ((1 - \acute{u}_1^l)^{\aleph} + (1 - \acute{u}_2^l)^{\aleph} - 1)^{1/\aleph}, 1 - ((1 - \acute{u}_1^u)^{\aleph} + (1 - \acute{u}_2^u)^{\aleph} - 1)^{1/\aleph} \right], \right. \\ \left. \left[((\bar{a}_1^l)^{\aleph} + (\bar{a}_2^l)^{\aleph} - 1)^{1/\aleph}, ((\bar{a}_1^u)^{\aleph} + (\bar{a}_2^u)^{\aleph} - 1)^{1/\aleph} \right], \right. \\ \left. \left[((\tilde{v}_1^l)^{\aleph} + (\tilde{v}_2^l)^{\aleph} - 1)^{1/\aleph}, ((\tilde{v}_1^u)^{\aleph} + (\tilde{v}_2^u)^{\aleph} - 1)^{1/\aleph} \right] \right), \aleph < 0$$

$$\mathbb{E}_1 \otimes \mathbb{E}_2 = \left(\left[((\acute{u}_1^l)^{\aleph} + (\acute{u}_2^l)^{\aleph} - 1)^{1/\aleph}, ((\acute{u}_1^u)^{\aleph} + (\acute{u}_2^u)^{\aleph} - 1)^{1/\aleph} \right], \right. \\ \left[1 - ((1 - \bar{a}_1^l)^{\aleph} + (1 - \bar{a}_2^l)^{\aleph} - 1)^{1/\aleph}, 1 - ((1 - \bar{a}_1^u)^{\aleph} + (1 - \bar{a}_2^u)^{\aleph} - 1)^{1/\aleph} \right], \\ \left[1 - ((1 - \tilde{v}_1^l)^{\aleph} + (1 - \tilde{v}_2^l)^{\aleph} - 1)^{1/\aleph}, 1 - ((1 - \tilde{v}_1^u)^{\aleph} + (1 - \tilde{v}_2^u)^{\aleph} - 1)^{1/\aleph} \right] \right), \aleph < 0$$

$$\mathfrak{S}\mathbb{E} = \left(\left[1 - (\mathfrak{S}((1 - \acute{u}^l)^{\aleph} - (\mathfrak{S} - 1)))^{1/\aleph}, 1 - (\mathfrak{S}((1 - \acute{u}^u)^{\aleph} - (\mathfrak{S} - 1)))^{1/\aleph} \right], \right. \\ \left[(\mathfrak{S}(\bar{a}^l)^{\aleph} - (\mathfrak{S} - 1))^{1/\aleph}, (\mathfrak{S}(\bar{a}^u)^{\aleph} - (\mathfrak{S} - 1))^{1/\aleph} \right], \\ \left[(\mathfrak{S}(\tilde{v}^l)^{\aleph} - (\mathfrak{S} - 1))^{1/\aleph}, (\mathfrak{S}(\tilde{v}^u)^{\aleph} - (\mathfrak{S} - 1))^{1/\aleph} \right] \right), \aleph < 0, \mathfrak{S} > 0$$

$$\mathbb{E}^{\mathfrak{S}} = \left(\left[(\mathfrak{S}(\acute{u}^l)^{\aleph} - (\mathfrak{S} - 1))^{1/\aleph}, (\mathfrak{S}(\acute{u}^u)^{\aleph} - (\mathfrak{S} - 1))^{1/\aleph} \right], \right. \\ \left[1 - (\mathfrak{S}((1 - \bar{a}^l)^{\aleph} - (\mathfrak{S} - 1)))^{1/\aleph}, 1 - (\mathfrak{S}((1 - \bar{a}^u)^{\aleph} - (\mathfrak{S} - 1)))^{1/\aleph} \right], \\ \left[1 - (\mathfrak{S}((1 - \tilde{v}^l)^{\aleph} - (\mathfrak{S} - 1)))^{1/\aleph}, 1 - (\mathfrak{S}((1 - \tilde{v}^u)^{\aleph} - (\mathfrak{S} - 1)))^{1/\aleph} \right] \right), \aleph < 0, \mathfrak{S} > 0$$

Theorem 1: Assume that $\mathbb{E}_1 = ([\acute{u}_{\mathbb{E}_1}^l, \acute{u}_{\mathbb{E}_1}^u], [\bar{a}_{\mathbb{E}_1}^l, \bar{a}_{\mathbb{E}_1}^u], [\tilde{v}_{\mathbb{E}_1}^l, \tilde{v}_{\mathbb{E}_1}^u])$ and $\mathbb{E}_2 = ([\acute{u}_{\mathbb{E}_2}^l, \acute{u}_{\mathbb{E}_2}^u], [\bar{a}_{\mathbb{E}_2}^l, \bar{a}_{\mathbb{E}_2}^u], [\tilde{v}_{\mathbb{E}_2}^l, \tilde{v}_{\mathbb{E}_2}^u])$ be any two IV-PFVs, and $\aleph < 0$ then the following axioms are listed below

1. $\mathbb{E}_1 \oplus \mathbb{E}_2 = \mathbb{E}_2 \oplus \mathbb{E}_1$
2. $\mathbb{E}_1 \otimes \mathbb{E}_2 = \mathbb{E}_2 \otimes \mathbb{E}_1$
3. $\hat{\mathbb{C}}(\mathbb{E}_1 \oplus \mathbb{E}_2) = \hat{\mathbb{C}}\mathbb{E}_1 \oplus \hat{\mathbb{C}}\mathbb{E}_2, \hat{\mathbb{C}} \geq 0$
4. $\hat{\mathbb{C}}_1 \mathbb{E}_1 \oplus \hat{\mathbb{C}}_2 \mathbb{E}_1 = (\hat{\mathbb{C}}_1 + \hat{\mathbb{C}}_2) \mathbb{E}_1, \hat{\mathbb{C}}_1, \hat{\mathbb{C}}_2 \geq 0$
5. $\mathbb{E}_1^{\hat{\mathbb{C}}_1} \otimes \mathbb{E}_1^{\hat{\mathbb{C}}_2} = \mathbb{E}_1^{\hat{\mathbb{C}}_1 + \hat{\mathbb{C}}_2}, \hat{\mathbb{C}}_1, \hat{\mathbb{C}}_2 \geq 0$
6. $\mathbb{E}_1^{\hat{\mathbb{C}}_1} \otimes \mathbb{E}_2^{\hat{\mathbb{C}}_1} = (\mathbb{E}_1 \otimes \mathbb{E}_2)^{\hat{\mathbb{C}}_1}, \hat{\mathbb{C}}_1 \geq 0$

Proof: Assume that, $\mathbb{E}_1 = ([\acute{u}_{\mathbb{E}_1}^l, \acute{u}_{\mathbb{E}_1}^u], [\bar{a}_{\mathbb{E}_1}^l, \bar{a}_{\mathbb{E}_1}^u], [\tilde{v}_{\mathbb{E}_1}^l, \tilde{v}_{\mathbb{E}_1}^u])$ and $\mathbb{E}_2 = ([\acute{u}_{\mathbb{E}_2}^l, \acute{u}_{\mathbb{E}_2}^u], [\bar{a}_{\mathbb{E}_2}^l, \bar{a}_{\mathbb{E}_2}^u], [\tilde{v}_{\mathbb{E}_2}^l, \tilde{v}_{\mathbb{E}_2}^u])$ are two IV-PFVs and $\hat{\mathbb{C}}_1, \hat{\mathbb{C}}_2 > 0$, then

1. $\mathbb{E}_1 \oplus \mathbb{E}_2$

$$\begin{aligned}
&= \left(\begin{array}{c} \left[1 - ((1 - \acute{u}_1^l)^{\mathfrak{N}} + (1 - \acute{u}_2^l)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}}, 1 - ((1 - \acute{u}_1^u)^{\mathfrak{N}} + (1 - \acute{u}_2^u)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}} \right], \\ \left[((\bar{a}_1^l)^{\mathfrak{N}} + (\bar{a}_2^l)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}}, ((\bar{a}_1^u)^{\mathfrak{N}} + (\bar{a}_2^u)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}} \right], \\ \left[((\tilde{v}_1^l)^{\mathfrak{N}} + (\tilde{v}_2^l)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}}, ((\tilde{v}_1^u)^{\mathfrak{N}} + (\tilde{v}_2^u)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}} \right] \end{array} \right) \\
&= \left(\begin{array}{c} \left[1 - ((1 - \acute{u}_2^l)^{\mathfrak{N}} + (1 - \acute{u}_1^l)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}}, 1 - ((1 - \acute{u}_2^u)^{\mathfrak{N}} + (1 - \acute{u}_1^u)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}} \right], \\ \left[((\bar{a}_2^l)^{\mathfrak{N}} + (\bar{a}_1^l)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}}, ((\bar{a}_2^u)^{\mathfrak{N}} + (\bar{a}_1^u)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}} \right], \\ \left[((\tilde{v}_2^l)^{\mathfrak{N}} + (\tilde{v}_1^l)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}}, ((\tilde{v}_2^u)^{\mathfrak{N}} + (\tilde{v}_1^u)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}} \right] \end{array} \right) \\
&= \mathbb{E}_2 \oplus \mathbb{E}_1
\end{aligned}$$

2. $\mathbb{E}_1 \otimes \mathbb{E}_2$

$$\begin{aligned}
&= \left(\begin{array}{c} \left[((\acute{u}_1^l)^{\mathfrak{N}} + (\acute{u}_2^l)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}}, ((\acute{u}_1^u)^{\mathfrak{N}} + (\acute{u}_2^u)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}} \right], \\ \left[1 - ((1 - \bar{a}_1^l)^{\mathfrak{N}} + (1 - \bar{a}_2^l)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}}, 1 - ((1 - \bar{a}_1^u)^{\mathfrak{N}} + (1 - \bar{a}_2^u)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}} \right], \\ \left[1 - ((1 - \tilde{v}_1^l)^{\mathfrak{N}} + (1 - \tilde{v}_2^l)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}}, 1 - ((1 - \tilde{v}_1^u)^{\mathfrak{N}} + (1 - \tilde{v}_2^u)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}} \right] \end{array} \right) \\
&= \left(\begin{array}{c} \left[((\acute{u}_2^l)^{\mathfrak{N}} + (\acute{u}_1^l)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}}, ((\acute{u}_2^u)^{\mathfrak{N}} + (\acute{u}_1^u)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}} \right], \\ \left[1 - ((1 - \bar{a}_2^l)^{\mathfrak{N}} + (1 - \bar{a}_1^l)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}}, 1 - ((1 - \bar{a}_2^u)^{\mathfrak{N}} + (1 - \bar{a}_1^u)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}} \right], \\ \left[1 - ((1 - \tilde{v}_2^l)^{\mathfrak{N}} + (1 - \tilde{v}_1^l)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}}, 1 - ((1 - \tilde{v}_2^u)^{\mathfrak{N}} + (1 - \tilde{v}_1^u)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}} \right] \end{array} \right) \\
&= \mathbb{E}_2 \otimes \mathbb{E}_1
\end{aligned}$$

3. $\hat{\mathbb{C}}(\mathbb{E}_1 \oplus \mathbb{E}_2)$

$$= \hat{\mathbb{C}} \left(\begin{array}{c} \left[1 - ((1 - \acute{u}_1^l)^{\mathfrak{N}} + (1 - \acute{u}_2^l)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}}, 1 - ((1 - \acute{u}_1^u)^{\mathfrak{N}} + (1 - \acute{u}_2^u)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}} \right], \\ \left[((\bar{a}_1^l)^{\mathfrak{N}} + (\bar{a}_2^l)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}}, ((\bar{a}_1^u)^{\mathfrak{N}} + (\bar{a}_2^u)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}} \right], \\ \left[((\tilde{v}_1^l)^{\mathfrak{N}} + (\tilde{v}_2^l)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}}, ((\tilde{v}_1^u)^{\mathfrak{N}} + (\tilde{v}_2^u)^{\mathfrak{N}} - 1)^{1/\mathfrak{N}} \right] \end{array} \right)$$

$$= \hat{\mathcal{C}}\mathbb{E}_1 \oplus \hat{\mathcal{C}}\mathbb{E}_2$$

$$\begin{aligned}
&= \left(\begin{pmatrix} \left[1 - \left(\tilde{\mathfrak{C}} \left((1 - \hat{u}_1^l)^{\mathfrak{N}} - (\tilde{\mathfrak{C}} - 1) \right) \right)^{1/\mathfrak{N}}, 1 - \left(\tilde{\mathfrak{C}} \left((1 - \hat{u}_1^u)^{\mathfrak{N}} - (\tilde{\mathfrak{C}} - 1) \right) \right)^{1/\mathfrak{N}} \right], \\ \left[\left(\tilde{\mathfrak{C}} (\bar{\mathfrak{a}}_1^l)^{\mathfrak{N}} - (\tilde{\mathfrak{C}} - 1) \right)^{1/\mathfrak{N}}, \left(\tilde{\mathfrak{C}} (\bar{\mathfrak{a}}_1^u)^{\mathfrak{N}} - (\tilde{\mathfrak{C}} - 1) \right)^{1/\mathfrak{N}} \right], \\ \left[\left(\tilde{\mathfrak{C}} (\tilde{\mathfrak{v}}_1^l)^{\mathfrak{N}} - (\tilde{\mathfrak{C}} - 1) \right)^{1/\mathfrak{N}}, \left(\tilde{\mathfrak{C}} (\tilde{\mathfrak{v}}_1^u)^{\mathfrak{N}} - (\tilde{\mathfrak{C}} - 1) \right)^{1/\mathfrak{N}} \right] \end{pmatrix} \right) \\
&\oplus \left(\begin{pmatrix} \left[1 - \left(\tilde{\mathfrak{C}} \left((1 - \hat{u}_2^l)^{\mathfrak{N}} - (\tilde{\mathfrak{C}} - 1) \right) \right)^{1/\mathfrak{N}}, 1 - \left(\tilde{\mathfrak{C}} \left((1 - \hat{u}_2^u)^{\mathfrak{N}} - (\tilde{\mathfrak{C}} - 1) \right) \right)^{1/\mathfrak{N}} \right], \\ \left[\left(\tilde{\mathfrak{C}} (\bar{\mathfrak{a}}_2^l)^{\mathfrak{N}} - (\tilde{\mathfrak{C}} - 1) \right)^{1/\mathfrak{N}}, \left(\tilde{\mathfrak{C}} (\bar{\mathfrak{a}}_2^u)^{\mathfrak{N}} - (\tilde{\mathfrak{C}} - 1) \right)^{1/\mathfrak{N}} \right], \\ \left[\left(\tilde{\mathfrak{C}} (\tilde{\mathfrak{v}}_2^l)^{\mathfrak{N}} - (\tilde{\mathfrak{C}} - 1) \right)^{1/\mathfrak{N}}, \left(\tilde{\mathfrak{C}} (\tilde{\mathfrak{v}}_2^u)^{\mathfrak{N}} - (\tilde{\mathfrak{C}} - 1) \right)^{1/\mathfrak{N}} \right] \end{pmatrix} \right) \\
&= \left(\begin{pmatrix} \left[1 - \left(\tilde{\mathfrak{C}} \left(\left(\left((1 - \hat{u}_1^l)^{\mathfrak{N}} + (1 - \hat{u}_2^l)^{\mathfrak{N}} - 1 \right)^{1/\mathfrak{N}} \right)^{\mathfrak{N}} - (\tilde{\mathfrak{C}} - 1) \right) \right)^{1/\mathfrak{N}}, \right. \\ \left. 1 - \left(\tilde{\mathfrak{C}} \left(\left((1 - \hat{u}_1^u)^{\mathfrak{N}} + (1 - \hat{u}_2^u)^{\mathfrak{N}} - 1 \right)^{1/\mathfrak{N}} \right)^{\mathfrak{N}} - (\tilde{\mathfrak{C}} - 1) \right) \right)^{1/\mathfrak{N}} \right], \\ \left[\left(\tilde{\mathfrak{C}} \left(\left((\bar{\mathfrak{a}}_1^l)^{\mathfrak{N}} + (\bar{\mathfrak{a}}_2^l)^{\mathfrak{N}} - 1 \right)^{1/\mathfrak{N}} \right)^{\mathfrak{N}} - (\tilde{\mathfrak{C}} - 1) \right)^{1/\mathfrak{N}}, \right. \\ \left[\left(\tilde{\mathfrak{C}} \left(\left((\bar{\mathfrak{a}}_1^u)^{\mathfrak{N}} + (\bar{\mathfrak{a}}_2^u)^{\mathfrak{N}} - 1 \right)^{1/\mathfrak{N}} \right)^{\mathfrak{N}} - (\tilde{\mathfrak{C}} - 1) \right)^{1/\mathfrak{N}} \right], \\ \left[\left(\tilde{\mathfrak{C}} \left(\left((\tilde{\mathfrak{v}}_1^l)^{\mathfrak{N}} + (\tilde{\mathfrak{v}}_2^l)^{\mathfrak{N}} - 1 \right)^{1/\mathfrak{N}} \right)^{\mathfrak{N}} - (\tilde{\mathfrak{C}} - 1) \right)^{1/\mathfrak{N}}, \right. \\ \left. \left[\left(\tilde{\mathfrak{C}} \left(\left((\tilde{\mathfrak{v}}_1^u)^{\mathfrak{N}} + (\tilde{\mathfrak{v}}_2^u)^{\mathfrak{N}} - 1 \right)^{1/\mathfrak{N}} \right)^{\mathfrak{N}} - (\tilde{\mathfrak{C}} - 1) \right)^{1/\mathfrak{N}} \right] \right] \end{pmatrix} \right)
\end{aligned}$$

$$= \left(\begin{array}{c} \left[1 - \left(\tilde{\omega} \left(\left((1 - \hat{u}_1^l)^{\omega} + (1 - \hat{u}_2^l)^{\omega} - 1 \right) \right) - (\tilde{\omega} - 1) \right) \right]^{1/\omega}, \\ \left[1 - \left(\tilde{\omega} \left(\left((1 - \hat{u}_1^u)^{\omega} + (1 - \hat{u}_2^u)^{\omega} - 1 \right)^{1/\omega} \right)^{\omega} - (\tilde{\omega} - 1) \right) \right]^{1/\omega}, \\ \left[\left(\tilde{\omega} \left(\left((\tilde{a}_1^l)^{\omega} + (\tilde{a}_2^l)^{\omega} - 1 \right)^{1/\omega} \right)^{\omega} - (\tilde{\omega} - 1) \right) \right]^{1/\omega}, \\ \left[\left(\tilde{\omega} \left(\left((\tilde{a}_1^u)^{\omega} + (\tilde{a}_2^u)^{\omega} - 1 \right)^{1/\omega} \right)^{\omega} - (\tilde{\omega} - 1) \right) \right]^{1/\omega}, \\ \left[\left(\tilde{\omega} \left(\left((\tilde{v}_1^l)^{\omega} + (\tilde{v}_2^l)^{\omega} - 1 \right)^{1/\omega} \right)^{\omega} - (\tilde{\omega} - 1) \right) \right]^{1/\omega}, \\ \left[\left(\tilde{\omega} \left(\left((\tilde{v}_1^u)^{\omega} + (\tilde{v}_2^u)^{\omega} - 1 \right)^{1/\omega} \right)^{\omega} - (\tilde{\omega} - 1) \right) \right]^{1/\omega} \end{array} \right)$$

$$= \left(\begin{array}{c} \left[1 - \left((\tilde{\omega}(1 - \hat{u}_1^l)^{\omega} + \tilde{\omega}(1 - \hat{u}_2^l)^{\omega} - 2\tilde{\omega} + 1) \right) \right]^{1/\omega}, \\ \left[1 - \left((\tilde{\omega}(1 - \hat{u}_1^u)^{\omega} + \tilde{\omega}(1 - \hat{u}_2^u)^{\omega} - 2\tilde{\omega} + 1) \right) \right]^{1/\omega}, \\ \left[\left((\tilde{\omega}(\tilde{a}_1^l)^{\omega} + \tilde{\omega}(\tilde{a}_2^l)^{\omega} - 2\tilde{\omega} + 1) \right) \right]^{1/\omega}, \\ \left[\left((\tilde{\omega}(\tilde{a}_1^u)^{\omega} + \tilde{\omega}(\tilde{a}_2^u)^{\omega} - 2\tilde{\omega} + 1) \right) \right]^{1/\omega}, \\ \left[\left((\tilde{\omega}(\tilde{v}_1^l)^{\omega} + \tilde{\omega}(\tilde{v}_2^l)^{\omega} - 2\tilde{\omega} + 1) \right) \right]^{1/\omega}, \\ \left[\left((\tilde{\omega}(\tilde{v}_1^u)^{\omega} + \tilde{\omega}(\tilde{v}_2^u)^{\omega} - 2\tilde{\omega} + 1) \right) \right]^{1/\omega} \end{array} \right)$$

4. $(\tilde{\omega}_1 + \tilde{\omega}_2)\tilde{E}_1$

$$\left(\begin{array}{c} \left[1 - \left((\tilde{\omega}_1 + \tilde{\omega}_2) \left((1 - \hat{u}_1^l)^{\omega} - (\tilde{\omega}_1 + \tilde{\omega}_2 - 1) \right) \right) \right]^{1/\omega}, \\ \left[1 - \left((\tilde{\omega}_1 + \tilde{\omega}_2) \left((1 - \hat{u}_1^u)^{\omega} - (\tilde{\omega}_1 + \tilde{\omega}_2 - 1) \right) \right) \right]^{1/\omega}, \\ \left[\left((\tilde{\omega}_1 + \tilde{\omega}_2)(\tilde{a}_1^l)^{\omega} - (\tilde{\omega}_1 + \tilde{\omega}_2 - 1) \right) \right]^{1/\omega}, \\ \left[\left((\tilde{\omega}_1 + \tilde{\omega}_2)(\tilde{a}_2^u)^{\omega} - (\tilde{\omega}_1 + \tilde{\omega}_2 - 1) \right) \right]^{1/\omega}, \\ \left[\left((\tilde{\omega}_1 + \tilde{\omega}_2)(\tilde{v}_1^l)^{\omega} - (\tilde{\omega}_1 + \tilde{\omega}_2 - 1) \right) \right]^{1/\omega}, \\ \left[\left((\tilde{\omega}_1 + \tilde{\omega}_2)(\tilde{v}_2^u)^{\omega} - (\tilde{\omega}_1 + \tilde{\omega}_2 - 1) \right) \right]^{1/\omega} \end{array} \right)$$

$$\left(\left(\begin{bmatrix} 1 - \left(\tilde{\omega}_1 \left((1 - \hat{u}_1^l)^{\omega} - (\tilde{\omega}_1 - 1) \right) \right)^{1/\omega}, \\ 1 - \left(\tilde{\omega}_1 \left((1 - \hat{u}_1^u)^{\omega} - (\tilde{\omega}_1 - 1) \right) \right)^{1/\omega}, \\ \left[\left(\tilde{\omega}_1 (\hat{a}_1^l)^{\omega} - (\tilde{\omega}_1 - 1) \right)^{1/\omega}, \right. \\ \left. \left[\left(\tilde{\omega}_1 (\hat{a}_2^u)^{\omega} - (\tilde{\omega}_1 - 1) \right)^{1/\omega} \right] \right], \\ \left[\left(\tilde{\omega}_1 (\hat{v}_1^l)^{\omega} - (\tilde{\omega}_1 - 1) \right)^{1/\omega}, \right. \\ \left. \left[\left(\tilde{\omega}_1 (\hat{v}_2^u)^{\omega} - (\tilde{\omega}_1 - 1) \right)^{1/\omega} \right] \right] \end{bmatrix} \right) \oplus \left(\begin{bmatrix} 1 - \left(\tilde{\omega}_2 \left((1 - \hat{u}_1^l)^{\omega} - (\tilde{\omega}_2 - 1) \right) \right)^{1/\omega}, \\ 1 - \left(\tilde{\omega}_2 \left((1 - \hat{u}_1^u)^{\omega} - (\tilde{\omega}_2 - 1) \right) \right)^{1/\omega}, \\ \left[\left(\tilde{\omega}_2 (\hat{a}_1^l)^{\omega} - (\tilde{\omega}_2 - 1) \right)^{1/\omega}, \right. \\ \left[\left(\tilde{\omega}_2 (\hat{a}_2^u)^{\omega} - (\tilde{\omega}_2 - 1) \right)^{1/\omega} \right] \right], \\ \left[\left(\tilde{\omega}_2 (\hat{v}_1^l)^{\omega} - (\tilde{\omega}_2 - 1) \right)^{1/\omega}, \right. \\ \left. \left[\left(\tilde{\omega}_2 (\hat{v}_2^u)^{\omega} - (\tilde{\omega}_2 - 1) \right)^{1/\omega} \right] \right] \end{bmatrix} \right) \right)$$

$$= \tilde{\omega}_1 \mathbb{E}_1 \oplus \tilde{\omega}_2 \mathbb{E}_1$$

5. This property proof $(\mathbb{E}_1^{\tilde{\omega}_1} \otimes \mathbb{E}_1^{\tilde{\omega}_2} = \mathbb{E}_1^{\tilde{\omega}_1 + \tilde{\omega}_2})$ is similar as $\tilde{\omega}_1 \mathbb{E}_1 \oplus \tilde{\omega}_2 \mathbb{E}_1 = (\tilde{\omega}_1 + \tilde{\omega}_2) \mathbb{E}_1$.

6. This property proof $(\mathbb{E}_1^{\tilde{\omega}_1} \otimes \mathbb{E}_2^{\tilde{\omega}_1} = (\mathbb{E}_1 \otimes \mathbb{E}_2)^{\tilde{\omega}_1})$ is similar to $\tilde{\omega}(\mathbb{E}_1 \oplus \mathbb{E}_2) = \tilde{\omega} \mathbb{E}_1 \oplus \tilde{\omega} \mathbb{E}_2$.

Definition 9: Assume that $\mathbb{E}_{\wp}$, $(\wp = 1, 2, \dots, \sigma)$ be the set of IV-PFVS. Then IV-PFSSPA is defined as:

$$IV - PFSSPA(\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3 \dots \mathbb{E}_{\sigma}) = \bigoplus_{\wp=1}^{\sigma} \frac{(1 + \mathbb{T}(\mathbb{E}_{\wp}))}{\sum_{\wp=1}^{\sigma} (1 + \mathbb{T}(\mathbb{E}_{\wp}))} \mathbb{E}_{\wp}, \quad (7)$$

where $\mathbb{T}(\mathbb{E}_{\wp}) = \sum_{\wp \neq \bar{\omega}}^{\sigma} S(\mathbb{E}_{\wp}, \mathbb{E}_{\bar{\omega}})$ and $S(\mathbb{E}_{\wp}, \mathbb{E}_{\bar{\omega}})$ represent the support value of $\mathbb{E}_{\wp}$ and $\mathbb{E}_{\bar{\omega}}$.

Theorem 2: Assume that $\mathbb{E}_{\wp} = ([\hat{u}_{\mathbb{E}_{\wp}}^l, \hat{u}_{\mathbb{E}_{\wp}}^u], [\hat{a}_{\mathbb{E}_{\wp}}^l, \hat{a}_{\mathbb{E}_{\wp}}^u], [\hat{v}_{\mathbb{E}_{\wp}}^l, \hat{v}_{\mathbb{E}_{\wp}}^u])$, $(\wp = 1, 2, \dots, \sigma)$ be the collection of IV-PFVs and $\omega < 0$ then the accumulative value of IV-PFV is also an IV-PFV given by:

$$IV - PFSSPA(\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3 \dots \mathbb{E}_{\sigma}) = \bigoplus_{\wp=1}^{\sigma} (\mathfrak{H}_{\wp} \mathbb{E}_{\wp}) \quad (8)$$

$$= \left(\begin{bmatrix} 1 - \left(\sum_{\wp=1}^{\sigma} \mathfrak{H}_{\wp} (1 - \hat{u}_{\mathbb{E}_{\wp}}^l)^{\omega} \right)^{1/\omega}, 1 - \left(\sum_{\wp=1}^{\sigma} \mathfrak{H}_{\wp} (1 - \hat{u}_{\mathbb{E}_{\wp}}^u)^{\omega} \right)^{1/\omega} \right], \\ \left[\left(\sum_{\wp=1}^{\sigma} \mathfrak{H}_{\wp} (\hat{a}_{\mathbb{E}_{\wp}}^l)^{\omega} \right)^{1/\omega}, \left(\sum_{\wp=1}^{\sigma} \mathfrak{H}_{\wp} (\hat{a}_{\mathbb{E}_{\wp}}^u)^{\omega} \right)^{1/\omega} \right], \\ \left[\left(\sum_{\wp=1}^{\sigma} \mathfrak{H}_{\wp} (\hat{v}_{\mathbb{E}_{\wp}}^l)^{\omega} \right)^{1/\omega}, \left(\sum_{\wp=1}^{\sigma} \mathfrak{H}_{\wp} (\hat{v}_{\mathbb{E}_{\wp}}^u)^{\omega} \right)^{1/\omega} \right] \end{bmatrix} \right),$$

where $\mathfrak{H}_{\wp} = \frac{(1 + \mathbb{T}(\mathbb{E}_{\wp})) \mathbb{E}_{\wp}}{\sum_{\wp=1}^{\sigma} (1 + \mathbb{T}(\mathbb{E}_{\wp}))}$, $(\wp = 1, 2, \dots, \sigma)$ and $\mathbb{T}(\mathbb{E}_{\wp}) = \sum_{\wp \neq k}^n S(\mathbb{E}_{\wp}, \mathbb{E}_{\bar{\omega}})$.

Proof:

All values of $\mathfrak{H} = (\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3, \dots, \mathfrak{H}_{\sigma})$,

$$\begin{aligned} \mathfrak{h}_1 \mathbb{E}_1 &= \left(\begin{array}{c} \left[1 - (\mathfrak{h}_1(1 - \mathfrak{u}_1^l))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, 1 - (\mathfrak{h}_1(1 - \mathfrak{u}_1^u))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, \\ \left[(\mathfrak{h}_1(\bar{\mathfrak{a}}_1^l))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, (\mathfrak{h}_1(\bar{\mathfrak{a}}_1^u))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, \left[(\mathfrak{h}_1(\tilde{\mathfrak{v}}_1^l))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, (\mathfrak{h}_1(\tilde{\mathfrak{v}}_1^u))^{\mathfrak{N}} \right]^{1/\mathfrak{N}} \end{array} \right), \\ \mathfrak{h}_2 \mathbb{E}_2 &= \left(\begin{array}{c} \left[1 - (\mathfrak{h}_2(1 - \mathfrak{u}_2^l))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, 1 - (\mathfrak{h}_2(1 - \mathfrak{u}_2^u))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, \\ \left[(\mathfrak{h}_2(\bar{\mathfrak{a}}_2^l))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, (\mathfrak{h}_2(\bar{\mathfrak{a}}_2^u))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, \left[(\mathfrak{h}_2(\tilde{\mathfrak{v}}_2^l))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, (\mathfrak{h}_2(\tilde{\mathfrak{v}}_2^u))^{\mathfrak{N}} \right]^{1/\mathfrak{N}} \end{array} \right) \end{aligned}$$

Now applying Definition 9, Then

$$\begin{aligned} IV - PFSSPA(\mathfrak{h}_1 \mathbb{E}_1 + \mathfrak{h}_2 \mathbb{E}_2) &= \left(\begin{array}{c} \left(\begin{array}{c} \left[1 - (\mathfrak{h}_1(1 - \mathfrak{u}_1^l))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, 1 - (\mathfrak{h}_1(1 - \mathfrak{u}_1^u))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, \\ \left[(\mathfrak{h}_1(\bar{\mathfrak{a}}_1^l))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, (\mathfrak{h}_1(\bar{\mathfrak{a}}_1^u))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, \\ \left[(\mathfrak{h}_1(\tilde{\mathfrak{v}}_1^l))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, (\mathfrak{h}_1(\tilde{\mathfrak{v}}_1^u))^{\mathfrak{N}} \right]^{1/\mathfrak{N}} \end{array} \right) \\ \oplus \left(\begin{array}{c} \left[1 - (\mathfrak{h}_2(1 - \mathfrak{u}_2^l))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, 1 - (\mathfrak{h}_2(1 - \mathfrak{u}_2^u))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, \\ \left[(\mathfrak{h}_2(\bar{\mathfrak{a}}_2^l))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, (\mathfrak{h}_2(\bar{\mathfrak{a}}_2^u))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, \\ \left[(\mathfrak{h}_2(\tilde{\mathfrak{v}}_2^l))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, (\mathfrak{h}_2(\tilde{\mathfrak{v}}_2^u))^{\mathfrak{N}} \right]^{1/\mathfrak{N}} \end{array} \right) \end{array} \right) \\ &= \left(\begin{array}{c} \left[1 - (\mathfrak{h}_1(1 - \mathfrak{u}_1^l))^{\mathfrak{N}} + \mathfrak{h}_2(1 - \mathfrak{u}_2^l))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, \\ \left[1 - (\mathfrak{h}_1(1 - \mathfrak{u}_1^u))^{\mathfrak{N}} + \mathfrak{h}_2(1 - \mathfrak{u}_2^u))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, \\ \left[(\mathfrak{h}_1(\bar{\mathfrak{a}}_1^l))^{\mathfrak{N}} + \mathfrak{h}_2(\bar{\mathfrak{a}}_2^l))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, (\mathfrak{h}_1(\bar{\mathfrak{a}}_1^u))^{\mathfrak{N}} + \mathfrak{h}_2(\bar{\mathfrak{a}}_2^u))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, \\ \left[(\mathfrak{h}_1(\tilde{\mathfrak{v}}_1^l))^{\mathfrak{N}} + \mathfrak{h}_2(\tilde{\mathfrak{v}}_2^l))^{\mathfrak{N}} \right]^{1/\mathfrak{N}}, (\mathfrak{h}_1(\tilde{\mathfrak{v}}_1^u))^{\mathfrak{N}} + \mathfrak{h}_2(\tilde{\mathfrak{v}}_2^u))^{\mathfrak{N}} \right]^{1/\mathfrak{N}} \end{array} \right) \\ &= \left(\begin{array}{c} \left[1 - \left(\sum_{\mathfrak{c}=1}^2 \mathfrak{h}_{\mathfrak{c}}(1 - \mathfrak{u}_{\mathfrak{c}}^l)^{\mathfrak{N}} \right)^{1/\mathfrak{N}}, 1 - \left(\sum_{\mathfrak{c}=1}^2 \mathfrak{h}_{\mathfrak{c}}(1 - \mathfrak{u}_{\mathfrak{c}}^u)^{\mathfrak{N}} \right)^{1/\mathfrak{N}} \right]^{1/\mathfrak{N}}, \\ \left[\left(\sum_{\mathfrak{c}=1}^2 \mathfrak{h}_{\mathfrak{c}}(\bar{\mathfrak{a}}_{\mathfrak{c}}^l)^{\mathfrak{N}} \right)^{1/\mathfrak{N}}, \left(\sum_{\mathfrak{c}=1}^2 \mathfrak{h}_{\mathfrak{c}}(\bar{\mathfrak{a}}_{\mathfrak{c}}^u)^{\mathfrak{N}} \right)^{1/\mathfrak{N}} \right]^{1/\mathfrak{N}}, \\ \left[\left(\sum_{\mathfrak{c}=1}^2 \mathfrak{h}_{\mathfrak{c}}(\tilde{\mathfrak{v}}_{\mathfrak{c}}^l)^{\mathfrak{N}} \right)^{1/\mathfrak{N}}, \left(\sum_{\mathfrak{c}=1}^2 \mathfrak{h}_{\mathfrak{c}}(\tilde{\mathfrak{v}}_{\mathfrak{c}}^u)^{\mathfrak{N}} \right)^{1/\mathfrak{N}} \right]^{1/\mathfrak{N}} \end{array} \right) \end{aligned}$$

The above outcome is true for $\mathfrak{c} = 2$.

Let us consider for $\mathfrak{c} = \sigma$, then

$$\begin{aligned}
IV - PFSSPA(\mathbb{E}_1, \mathbb{E}_2, \dots, \mathbb{E}_\sigma) &= \bigoplus_{\mathfrak{c}=1}^{\sigma} (\mathfrak{h}_{\mathfrak{c}} \mathbb{E}_{\mathfrak{c}}) \\
&= \left(\begin{array}{c} \left[1 - \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (1 - \mathfrak{u}_{\mathfrak{c}}^l)^{\mathfrak{q}} \right)^{1/\mathfrak{q}}, 1 - \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (1 - \mathfrak{u}_{\mathfrak{c}}^u)^{\mathfrak{q}} \right)^{1/\mathfrak{q}} \right], \\ \left[\left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\bar{\mathfrak{a}}_{\mathfrak{c}}^l)^{\mathfrak{q}} \right)^{1/\mathfrak{q}}, \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\bar{\mathfrak{a}}_{\mathfrak{c}}^u)^{\mathfrak{q}} \right)^{1/\mathfrak{q}} \right], \\ \left[\left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\tilde{\mathfrak{v}}_{\mathfrak{c}}^l)^{\mathfrak{q}} \right)^{1/\mathfrak{q}}, \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\tilde{\mathfrak{v}}_{\mathfrak{c}}^u)^{\mathfrak{q}} \right)^{1/\mathfrak{q}} \right] \end{array} \right)
\end{aligned}$$

If we take $\mathfrak{c} = \sigma + 1$, then we get

$$\begin{aligned}
IV - PFSSPA(\mathbb{E}_1, \mathbb{E}_2, \dots, \mathbb{E}_{\sigma}, \mathbb{E}_{\sigma+1}) &= \bigoplus_{\mathfrak{c}=1}^{\sigma} (\mathfrak{h}_{\mathfrak{c}} \mathbb{E}_{\mathfrak{c}}) \oplus (\mathfrak{h}_{\sigma+1} \mathbb{E}_{\sigma+1}) \\
&= \left(\begin{array}{c} \left(\begin{array}{c} \left[1 - \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (1 - \mathfrak{u}_{\mathfrak{c}}^l)^{\mathfrak{q}} \right)^{1/\mathfrak{q}}, 1 - \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (1 - \mathfrak{u}_{\mathfrak{c}}^u)^{\mathfrak{q}} \right)^{1/\mathfrak{q}} \right], \\ \left[\left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\bar{\mathfrak{a}}_{\mathfrak{c}}^l)^{\mathfrak{q}} \right)^{1/\mathfrak{q}}, \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\bar{\mathfrak{a}}_{\mathfrak{c}}^u)^{\mathfrak{q}} \right)^{1/\mathfrak{q}} \right], \\ \left[\left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\tilde{\mathfrak{v}}_{\mathfrak{c}}^l)^{\mathfrak{q}} \right)^{1/\mathfrak{q}}, \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\tilde{\mathfrak{v}}_{\mathfrak{c}}^u)^{\mathfrak{q}} \right)^{1/\mathfrak{q}} \right] \end{array} \right) \\ \oplus \left(\begin{array}{c} \left[1 - \left(\mathfrak{h}_{\sigma+1} (1 - \mathfrak{u}_{\sigma+1}^l)^{\mathfrak{q}} \right)^{1/\mathfrak{q}}, 1 - \left(\mathfrak{h}_{\sigma+1} (1 - \mathfrak{u}_{\sigma+1}^u)^{\mathfrak{q}} \right)^{1/\mathfrak{q}} \right], \\ \left[\left(\mathfrak{h}_{\sigma+1} (\bar{\mathfrak{a}}_{\sigma+1}^l)^{\mathfrak{q}} \right)^{1/\mathfrak{q}}, \left(\mathfrak{h}_{\sigma+1} (\bar{\mathfrak{a}}_{\sigma+1}^u)^{\mathfrak{q}} \right)^{1/\mathfrak{q}} \right], \\ \left[\left(\mathfrak{h}_{\sigma+1} (\tilde{\mathfrak{v}}_{\sigma+1}^l)^{\mathfrak{q}} \right)^{1/\mathfrak{q}}, \left(\mathfrak{h}_{\sigma+1} (\tilde{\mathfrak{v}}_{\sigma+1}^u)^{\mathfrak{q}} \right)^{1/\mathfrak{q}} \right] \end{array} \right) \end{array} \right) \\
&= \left(\begin{array}{c} \left[1 - \left(\sum_{\mathfrak{c}=1}^{\sigma+1} \mathfrak{h}_{\mathfrak{c}} (1 - \mathfrak{u}_{\mathfrak{c}}^l)^{\mathfrak{q}} \right)^{1/\mathfrak{q}}, 1 - \left(\sum_{\mathfrak{c}=1}^{\sigma+1} \mathfrak{h}_{\mathfrak{c}} (1 - \mathfrak{u}_{\mathfrak{c}}^u)^{\mathfrak{q}} \right)^{1/\mathfrak{q}} \right], \\ \left[\left(\sum_{\mathfrak{c}=1}^{\sigma+1} \mathfrak{h}_{\mathfrak{c}} (\bar{\mathfrak{a}}_{\mathfrak{c}}^l)^{\mathfrak{q}} \right)^{1/\mathfrak{q}}, \left(\sum_{\mathfrak{c}=1}^{\sigma+1} \mathfrak{h}_{\mathfrak{c}} (\bar{\mathfrak{a}}_{\mathfrak{c}}^u)^{\mathfrak{q}} \right)^{1/\mathfrak{q}} \right], \\ \left[\left(\sum_{\mathfrak{c}=1}^{\sigma+1} \mathfrak{h}_{\mathfrak{c}} (\tilde{\mathfrak{v}}_{\mathfrak{c}}^l)^{\mathfrak{q}} \right)^{1/\mathfrak{q}}, \left(\sum_{\mathfrak{c}=1}^{\sigma+1} \mathfrak{h}_{\mathfrak{c}} (\tilde{\mathfrak{v}}_{\mathfrak{c}}^u)^{\mathfrak{q}} \right)^{1/\mathfrak{q}} \right] \end{array} \right)
\end{aligned}$$

Hence, the result holds for $\mathfrak{c} = \sigma + 1$.

Therefore,

$$IV - PFSSPA(\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3 \dots \mathbb{E}_\sigma) = \left(\begin{array}{c} \left[1 - \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (1 - \mathfrak{u}_{\mathfrak{c}}^l)^{\mathfrak{N}} \right)^{1/\mathfrak{N}}, 1 - \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (1 - \mathfrak{u}_{\mathfrak{c}}^u)^{\mathfrak{N}} \right)^{1/\mathfrak{N}} \right], \\ \left[\left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\bar{\mathfrak{a}}_{\mathfrak{c}}^l)^{\mathfrak{N}} \right)^{1/\mathfrak{N}}, \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\bar{\mathfrak{a}}_{\mathfrak{c}}^u)^{\mathfrak{N}} \right)^{1/\mathfrak{N}} \right], \\ \left[\left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\tilde{\mathfrak{v}}_{\mathfrak{c}}^l)^{\mathfrak{N}} \right)^{1/\mathfrak{N}}, \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\tilde{\mathfrak{v}}_{\mathfrak{c}}^u)^{\mathfrak{N}} \right)^{1/\mathfrak{N}} \right] \end{array} \right)$$

Theorem 3: (Idempotency) Assume that $\mathbb{E}_{\mathfrak{c}} = ([\mathfrak{u}_{\mathfrak{c}}^l, \mathfrak{u}_{\mathfrak{c}}^u], [\bar{\mathfrak{a}}_{\mathfrak{c}}^l, \bar{\mathfrak{a}}_{\mathfrak{c}}^u], [\tilde{\mathfrak{v}}_{\mathfrak{c}}^l, \tilde{\mathfrak{v}}_{\mathfrak{c}}^u])$, $(\mathfrak{c} = 1, 2, \dots, \sigma)$ be the set of all identical IV-PFVs. If $\mathbb{E}_{\mathfrak{c}} = ([\mathfrak{u}_{\mathfrak{c}}^l, \mathfrak{u}_{\mathfrak{c}}^u], [\bar{\mathfrak{a}}_{\mathfrak{c}}^l, \bar{\mathfrak{a}}_{\mathfrak{c}}^u], [\tilde{\mathfrak{v}}_{\mathfrak{c}}^l, \tilde{\mathfrak{v}}_{\mathfrak{c}}^u])$ for all $\mathfrak{c} \in \{1, 2, \dots, \sigma\}$, then

$$IV - PFSSPA(\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3, \dots, \mathbb{E}_{\sigma}) = \mathbb{E} \quad (9)$$

Proof:

Since

$$\begin{aligned} & IV - PFSSPA(\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3 \dots \mathbb{E}_{\sigma}) \\ &= \left(\begin{array}{c} \left[1 - \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (1 - \mathfrak{u}_{\mathfrak{c}}^l)^{\mathfrak{N}} \right)^{1/\mathfrak{N}}, 1 - \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (1 - \mathfrak{u}_{\mathfrak{c}}^u)^{\mathfrak{N}} \right)^{1/\mathfrak{N}} \right], \\ \left[\left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\bar{\mathfrak{a}}_{\mathfrak{c}}^l)^{\mathfrak{N}} \right)^{1/\mathfrak{N}}, \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\bar{\mathfrak{a}}_{\mathfrak{c}}^u)^{\mathfrak{N}} \right)^{1/\mathfrak{N}} \right], \\ \left[\left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\tilde{\mathfrak{v}}_{\mathfrak{c}}^l)^{\mathfrak{N}} \right)^{1/\mathfrak{N}}, \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\tilde{\mathfrak{v}}_{\mathfrak{c}}^u)^{\mathfrak{N}} \right)^{1/\mathfrak{N}} \right] \end{array} \right) \\ &= \left(\begin{array}{c} \left[1 - \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (1 - \mathfrak{u}_{\mathfrak{c}}^l)^{\mathfrak{N}} \right)^{1/\mathfrak{N}}, 1 - \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (1 - \mathfrak{u}_{\mathfrak{c}}^u)^{\mathfrak{N}} \right)^{1/\mathfrak{N}} \right], \\ \left[\left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\bar{\mathfrak{a}}_{\mathfrak{c}}^l)^{\mathfrak{N}} \right)^{1/\mathfrak{N}}, \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\bar{\mathfrak{a}}_{\mathfrak{c}}^u)^{\mathfrak{N}} \right)^{1/\mathfrak{N}} \right], \\ \left[\left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\tilde{\mathfrak{v}}_{\mathfrak{c}}^l)^{\mathfrak{N}} \right)^{1/\mathfrak{N}}, \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\tilde{\mathfrak{v}}_{\mathfrak{c}}^u)^{\mathfrak{N}} \right)^{1/\mathfrak{N}} \right] \end{array} \right) \\ &= ([\mathfrak{u}^l, \mathfrak{u}^u], [\bar{\mathfrak{a}}^l, \bar{\mathfrak{a}}^u], [\tilde{\mathfrak{v}}^l, \tilde{\mathfrak{v}}^u]) = \mathbb{E} \end{aligned}$$

Theorem 4: (Monotonicity) Assume that $\mathbb{E}_{\mathfrak{c}} = ([\mathfrak{u}_{\mathfrak{c}}^l, \mathfrak{u}_{\mathfrak{c}}^u], [\bar{\mathfrak{a}}_{\mathfrak{c}}^l, \bar{\mathfrak{a}}_{\mathfrak{c}}^u], [\tilde{\mathfrak{v}}_{\mathfrak{c}}^l, \tilde{\mathfrak{v}}_{\mathfrak{c}}^u])$ and $\mathbb{E}'_{\mathfrak{c}} = ([\mathfrak{u}'_{\mathfrak{c}}^l, \mathfrak{u}'_{\mathfrak{c}}^u], [\bar{\mathfrak{a}}'_{\mathfrak{c}}^l, \bar{\mathfrak{a}}'_{\mathfrak{c}}^u], [\tilde{\mathfrak{v}}'_{\mathfrak{c}}^l, \tilde{\mathfrak{v}}'_{\mathfrak{c}}^u])$ be the collection of IV-PFVs, and $\mathfrak{u}_{\mathfrak{c}}^l \leq \mathfrak{u}'_{\mathfrak{c}}^l, \mathfrak{u}_{\mathfrak{c}}^u \leq \mathfrak{u}'_{\mathfrak{c}}^u, \bar{\mathfrak{a}}_{\mathfrak{c}}^l \leq \bar{\mathfrak{a}}'_{\mathfrak{c}}^l, \bar{\mathfrak{a}}_{\mathfrak{c}}^u \leq \bar{\mathfrak{a}}'_{\mathfrak{c}}^u, \tilde{\mathfrak{v}}_{\mathfrak{c}}^l \leq \tilde{\mathfrak{v}}'_{\mathfrak{c}}^l, \tilde{\mathfrak{v}}_{\mathfrak{c}}^u \leq \tilde{\mathfrak{v}}'_{\mathfrak{c}}^u$ for each $\mathfrak{c} \in \{1, 2, \dots, \sigma\}$, then

$$IV - PFSSPA(\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3 \dots \mathbb{E}_{\sigma}) \leq IV - PFSSPA(\mathbb{E}'_1, \mathbb{E}'_2, \mathbb{E}'_3 \dots \mathbb{E}'_{\sigma}) \quad (10)$$

Proof:

Since

$$IV - PFSSPA(\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3 \dots \mathbb{E}_\sigma) = \left(\begin{array}{c} \left[1 - \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (1 - \mathfrak{u}_{\mathfrak{c}}^l)^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}}, 1 - \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (1 - \mathfrak{u}_{\mathfrak{c}}^u)^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}} \right], \\ \left[\left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\bar{\mathfrak{a}}_{\mathfrak{c}}^l)^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}}, \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\bar{\mathfrak{a}}_{\mathfrak{c}}^u)^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}} \right], \\ \left[\left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\tilde{\mathfrak{v}}_{\mathfrak{c}}^l)^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}}, \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\tilde{\mathfrak{v}}_{\mathfrak{c}}^u)^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}} \right] \end{array} \right)$$

And

$$IV - PFSSPA(\mathbb{E}'_1, \mathbb{E}'_2, \mathbb{E}'_3 \dots \mathbb{E}'_\sigma) = \left(\begin{array}{c} \left[1 - \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}'_{\mathfrak{c}} (1 - \mathfrak{u}'_{\mathfrak{c}}^l)^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}}, 1 - \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}'_{\mathfrak{c}} (1 - \mathfrak{u}'_{\mathfrak{c}}^u)^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}} \right], \\ \left[\left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}'_{\mathfrak{c}} (\bar{\mathfrak{a}}'^l_{\mathfrak{c}})^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}}, \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}'_{\mathfrak{c}} (\bar{\mathfrak{a}}'^u_{\mathfrak{c}})^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}} \right], \\ \left[\left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}'_{\mathfrak{c}} (\tilde{\mathfrak{v}}'^l_{\mathfrak{c}})^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}}, \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}'_{\mathfrak{c}} (\tilde{\mathfrak{v}}'^u_{\mathfrak{c}})^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}} \right] \end{array} \right)$$

We know that $\mathfrak{u}_{\mathfrak{c}}^l \leq \mathfrak{u}'_{\mathfrak{c}}^l$ for each $\mathfrak{c} \in \{1, 2, \dots, \sigma\}$

$$\begin{aligned} (1 - \mathfrak{u}_{\mathfrak{c}}^l)^{\mathfrak{Q}} &\leq (1 - \mathfrak{u}'_{\mathfrak{c}}^l)^{\mathfrak{Q}}, \text{ since } \mathfrak{Q} < 0, \\ \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}'_{\mathfrak{c}} (1 - \mathfrak{u}'_{\mathfrak{c}}^l)^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}} &\leq \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (1 - \mathfrak{u}_{\mathfrak{c}}^l)^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}} \\ 1 - \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (1 - \mathfrak{u}_{\mathfrak{c}}^l)^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}} &\leq 1 - \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}'_{\mathfrak{c}} (1 - \mathfrak{u}'_{\mathfrak{c}}^l)^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}}. \end{aligned}$$

Similarly, if $\mathfrak{u}_{\mathfrak{c}}^u \leq \mathfrak{u}'_{\mathfrak{c}}^u$, $\bar{\mathfrak{a}}'^l_{\mathfrak{c}} \leq \bar{\mathfrak{a}}_{\mathfrak{c}}^l$, $\bar{\mathfrak{a}}'^u_{\mathfrak{c}} \leq \bar{\mathfrak{a}}_{\mathfrak{c}}^u$, $\tilde{\mathfrak{v}}'^l_{\mathfrak{c}} \leq \tilde{\mathfrak{v}}_{\mathfrak{c}}^l$, and $\tilde{\mathfrak{v}}'^u_{\mathfrak{c}} \leq \tilde{\mathfrak{v}}_{\mathfrak{c}}^u$, it can be demonstrated that

$$\begin{aligned} 1 - \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (1 - \mathfrak{u}_{\mathfrak{c}}^u)^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}} &\leq 1 - \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}'_{\mathfrak{c}} (1 - \mathfrak{u}'_{\mathfrak{c}}^u)^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}}, \\ \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}'_{\mathfrak{c}} (\bar{\mathfrak{a}}'^l_{\mathfrak{c}})^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}} &\leq \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\bar{\mathfrak{a}}_{\mathfrak{c}}^l)^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}}, \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}'_{\mathfrak{c}} (\bar{\mathfrak{a}}'^u_{\mathfrak{c}})^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}} \leq \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\bar{\mathfrak{a}}_{\mathfrak{c}}^u)^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}}, \\ \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}'_{\mathfrak{c}} (\tilde{\mathfrak{v}}'^l_{\mathfrak{c}})^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}} &\leq \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\tilde{\mathfrak{v}}_{\mathfrak{c}}^l)^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}}, \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}'_{\mathfrak{c}} (\tilde{\mathfrak{v}}'^u_{\mathfrak{c}})^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}} \leq \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{h}_{\mathfrak{c}} (\tilde{\mathfrak{v}}_{\mathfrak{c}}^u)^{\mathfrak{Q}} \right)^{1/\mathfrak{Q}}, \end{aligned}$$

Hence, $IV - PFSSPA(\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3 \dots \mathbb{E}_n) \leq IV - PFSSA(\mathbb{E}'_1, \mathbb{E}'_2, \mathbb{E}'_3 \dots \mathbb{E}'_\sigma)$.

Theorem 5: (Boundedness) Assume that $\mathbb{E}_{\mathfrak{c}} = ([\mathfrak{u}_{\mathbb{E}_{\mathfrak{c}}}^l, \mathfrak{u}_{\mathbb{E}_{\mathfrak{c}}}^u], [\bar{\mathfrak{a}}_{\mathbb{E}_{\mathfrak{c}}}^l, \bar{\mathfrak{a}}_{\mathbb{E}_{\mathfrak{c}}}^u], [\tilde{\mathfrak{v}}_{\mathbb{E}_{\mathfrak{c}}}^l, \tilde{\mathfrak{v}}_{\mathbb{E}_{\mathfrak{c}}}^u])$, ($\mathfrak{c} = 1, 2, \dots, \sigma$) be the collection of IV-PFVs, that is $\mathbb{E}^- = \min(\mathbb{E}_1, \mathbb{E}_2, \dots, \mathbb{E}_\sigma)$ and $\mathbb{E}^+ = \max(\mathbb{E}_1, \mathbb{E}_2, \dots, \mathbb{E}_\sigma)$ then:

$$\mathbb{E}^- \leq IV - PFSSPA(\mathbb{E}_1, \mathbb{E}_2, \dots, \mathbb{E}_\sigma) \leq \mathbb{E}^+. \quad (11)$$

Proof: Assume that $\mathbb{E}_\zeta = ([\hat{u}_{\mathbb{E}_\zeta}^l, \hat{u}_{\mathbb{E}_\zeta}^u], [\bar{a}_{\mathbb{E}_\zeta}^l, \bar{a}_{\mathbb{E}_\zeta}^u], [\tilde{v}_{\mathbb{E}_\zeta}^l, \tilde{v}_{\mathbb{E}_\zeta}^u])$ be the set of IV-PFVs. Let $\mathbb{E}^- = \min(\mathbb{E}_1, \mathbb{E}_2, \dots, \mathbb{E}_\sigma) = (\hat{u}_\zeta^-, \bar{a}_\zeta^-, \tilde{v}_\zeta^-)$ and $\mathbb{E}^+ = \max(\mathbb{E}_1, \mathbb{E}_2, \dots, \mathbb{E}_\sigma) = (\hat{u}_\zeta^+, \bar{a}_\zeta^+, \tilde{v}_\zeta^+)$. Then

$$\begin{aligned} & \left[1 - \left(\sum_{\zeta=1}^{\sigma} \eta_{\zeta} (1 - \hat{u}_{\zeta}^{l-})^{\omega} \right)^{1/\omega}, 1 - \left(\sum_{\zeta=1}^{\sigma} \eta_{\zeta} (1 - \hat{u}_{\zeta}^{u-})^{\omega} \right)^{1/\omega} \right] \\ & \leq \left[1 - \left(\sum_{\zeta=1}^{\sigma} \eta_{\zeta} (1 - \hat{u}_{\zeta}^l)^{\omega} \right)^{1/\omega}, 1 - \left(\sum_{\zeta=1}^{\sigma} \eta_{\zeta} (1 - \hat{u}_{\zeta}^u)^{\omega} \right)^{1/\omega} \right] \\ & \leq \left[1 - \left(\sum_{\zeta=1}^{\sigma} \eta_{\zeta} (1 - \hat{u}_{\zeta}^{l+})^{\omega} \right)^{1/\omega}, 1 - \left(\sum_{\zeta=1}^{\sigma} \eta_{\zeta} (1 - \hat{u}_{\zeta}^{u+})^{\omega} \right)^{1/\omega} \right], \\ & \left[\left(\sum_{\zeta=1}^{\sigma} \eta_{\zeta} (\bar{a}_{\zeta}^{l-})^{\omega} \right)^{1/\omega}, \left(\sum_{\zeta=1}^{\sigma} \eta_{\zeta} (\bar{a}_{\zeta}^{u-})^{\omega} \right)^{1/\omega} \right] \geq \left[\left(\sum_{\zeta=1}^{\sigma} \eta_{\zeta} (\bar{a}_{\zeta}^l)^{\omega} \right)^{1/\omega}, \left(\sum_{\zeta=1}^{\sigma} \eta_{\zeta} (\bar{a}_{\zeta}^u)^{\omega} \right)^{1/\omega} \right] \\ & \geq \left[\left(\sum_{\zeta=1}^{\sigma} \eta_{\zeta} (\bar{a}_{\zeta}^{l+})^{\omega} \right)^{1/\omega}, \left(\sum_{\zeta=1}^{\sigma} \eta_{\zeta} (\bar{a}_{\zeta}^{u+})^{\omega} \right)^{1/\omega} \right], \\ & \left[\left(\sum_{\zeta=1}^{\sigma} \eta_{\zeta} (\tilde{v}_{\zeta}^{l-})^{\omega} \right)^{1/\omega}, \left(\sum_{\zeta=1}^{\sigma} \eta_{\zeta} (\tilde{v}_{\zeta}^{u-})^{\omega} \right)^{1/\omega} \right] \geq \left[\left(\sum_{\zeta=1}^{\sigma} \eta_{\zeta} (\tilde{v}_{\zeta}^l)^{\omega} \right)^{1/\omega}, \left(\sum_{\zeta=1}^{\sigma} \eta_{\zeta} (\tilde{v}_{\zeta}^u)^{\omega} \right)^{1/\omega} \right] \\ & \geq \left[\left(\sum_{\zeta=1}^{\sigma} \eta_{\zeta} (\tilde{v}_{\zeta}^{l+})^{\omega} \right)^{1/\omega}, \left(\sum_{\zeta=1}^{\sigma} \eta_{\zeta} (\tilde{v}_{\zeta}^{u+})^{\omega} \right)^{1/\omega} \right], \end{aligned}$$

Therefore,

$$\mathbb{E}^- \leq IV - PFSSPA(\mathbb{E}_1, \mathbb{E}_2, \dots, \mathbb{E}_\sigma) \leq \mathbb{E}^+.$$

Definition 10: Assume that $\mathbb{E}_\zeta = ([\hat{u}_{\mathbb{E}_\zeta}^l, \hat{u}_{\mathbb{E}_\zeta}^u], [\bar{a}_{\mathbb{E}_\zeta}^l, \bar{a}_{\mathbb{E}_\zeta}^u], [\tilde{v}_{\mathbb{E}_\zeta}^l, \tilde{v}_{\mathbb{E}_\zeta}^u])$ ($\zeta = 1, 2, \dots, \sigma$) be the collection of IV-PFV with weight $\varepsilon_\zeta = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_\sigma)^T$ such that $\varepsilon_\zeta \in [0, 1]$ for $\zeta \in \{1, 2, \dots, \sigma\}$ and $\sum_{\zeta=1}^{\sigma} \varepsilon_\zeta = 1$. Then IV-PFSSPWA operator is

$$IV - PFSSPWA(\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3 \dots \mathbb{E}_\sigma) = \bigoplus_{\zeta=1}^{\sigma} \frac{(\varepsilon_\zeta (1 + \mathbb{T}(\mathbb{E}_\zeta)) \mathbb{E}_\zeta)}{\sum_{\zeta=1}^{\sigma} \varepsilon_\zeta (1 + \mathbb{T}(\mathbb{E}_\zeta))} \quad (12)$$

Where $\mathbb{T}(\mathbb{E}_\zeta) = \sum_{\zeta \neq \bar{\zeta}}^{\sigma} S(\mathbb{E}_\zeta, \mathbb{E}_{\bar{\zeta}})$ and $S(\mathbb{E}_\zeta, \mathbb{E}_{\bar{\zeta}})$ indicate the support value of $\mathbb{E}_\zeta$ and $\mathbb{E}_{\bar{\zeta}}$.

Example 1: Let $\mathbb{E}_1 = ([0.32, 0.34], [0.25, 0.28], [0.09, 0.1])$, $\mathbb{E}_2 = ([0.35, 0.38], [0.32, 0.37], [0.08, 0.15])$, $\mathbb{E}_3 = ([0.42, 0.45], [0.17, 0.25], [0.02, 0.22])$ and $\mathbb{E}_4 = ([0.18, 0.23], [0.41, 0.45], [0.07, 0.17])$ be the four IV-PFVs with weight vectors are $\varepsilon_\zeta = (0.23, 0.35, 0.10, 0.32)$. For $\omega = -1$, then applying Definition 12 to find out the required findings for IV-PFSSWPA operators as follows

Initially, we have to determine the $Supr(\mathbb{E}_\zeta, \mathbb{E}_{\bar{\zeta}})$

$$\check{s}(1,2) = 0.7350, \check{s}(1,3) = 0.5733, \check{s}(1,4) = 0.4467, \check{s}(2,3) = 0.5183, \check{s}(2,4) = 0.6217, \check{s}(3,4) = 0.2,$$

To calculate $T_{\lambda}(\mathbb{E}_{\mathfrak{c}}) = \sum_{\substack{\mathfrak{c}=1 \\ \mathfrak{c} \neq \mathfrak{w}}}^{\sigma} S(\mathbb{E}_{\mathfrak{c}}, \mathbb{E}_{\mathfrak{w}})$, then we have

$$T_1 = 1.755, T_2 = 1.875, T_3 = 1.292, T_4 = 1.268,$$

Next, to determine $\mathfrak{T}_{\mathfrak{c}}$

$$\mathfrak{T}_{\mathfrak{c}} = \frac{\varepsilon_{\mathfrak{c}}(T_{\lambda}(\mathbb{E}_{\mathfrak{c}}) + 1)}{\sum_{\mathfrak{y}=1}^{\sigma} \varepsilon_{\mathfrak{y}}(T_{\lambda}(\mathbb{E}_{\mathfrak{y}}) + 1)}$$

$$\mathfrak{T}_1 = 0.2442, \mathfrak{T}_2 = 0.3878, \mathfrak{T}_3 = 0.0883, \mathfrak{T}_4 = 0.2797$$

The accumulative outcomes for the IV-PFSSWPA operator are given as

$$\begin{aligned} & IV - PFSSWPA(\mathbb{E}_1, \mathbb{E}_2, \dots, \mathbb{E}_{\sigma}) \\ &= ([0.3099, 0.3418], [0.3227, 0.3675], [0.0747, 0.1509]) \end{aligned}$$

Theorem 6: Assume that $\mathbb{E}_{\mathfrak{c}} = ([\mathfrak{u}_{\mathbb{E}_{\mathfrak{c}}}^l, \mathfrak{u}_{\mathbb{E}_{\mathfrak{c}}}^u], [\mathfrak{a}_{\mathbb{E}_{\mathfrak{c}}}^l, \mathfrak{a}_{\mathbb{E}_{\mathfrak{c}}}^u], [\mathfrak{v}_{\mathbb{E}_{\mathfrak{c}}}^l, \mathfrak{v}_{\mathbb{E}_{\mathfrak{c}}}^u])$ ($\mathfrak{c} = 1, 2, \dots, \sigma$) be the collection of IV-PFVs and $\Omega < 0$ with weight vectors $\varepsilon_{\mathfrak{c}} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{\sigma})^T$ such that $\varepsilon_{\mathfrak{c}} \in [0, 1]$ for $\mathfrak{c} \in \{1, 2, \dots, \sigma\}$ and $\sum_{\mathfrak{c}=1}^{\sigma} \varepsilon_{\mathfrak{c}} = 1$. Then the accumulative consequence of the IV-PFSSWPA operator is also an IV-PFV and we have:

$$\begin{aligned} IV - PFSSWPA(\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3 \dots \mathbb{E}_{\sigma}) &= \bigoplus_{\mathfrak{c}=1}^{\sigma} (\varepsilon_{\mathfrak{c}} \mathbb{E}_{\mathfrak{c}}) = \bigoplus_{\mathfrak{c}=1}^{\sigma} \frac{(\varepsilon_{\mathfrak{c}}(1 + T_{\lambda}(\mathbb{E}_{\mathfrak{c}}))\mathbb{E}_{\mathfrak{c}})}{\sum_{\mathfrak{y}=1}^{\sigma} \varepsilon_{\mathfrak{y}}(1 + T_{\lambda}(\mathbb{E}_{\mathfrak{y}}))} \\ &= \left(\left[1 - \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{T}_{\mathfrak{c}} (1 - \mathfrak{u}^l)^{\Omega} \right)^{1/\Omega}, 1 - \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{T}_{\mathfrak{c}} (1 - \mathfrak{u}^u)^{\Omega} \right)^{1/\Omega} \right], \right. \\ &\quad \left[\left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{T}_{\mathfrak{c}} (\mathfrak{a}^l)^{\Omega} \right)^{1/\Omega}, \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{T}_{\mathfrak{c}} (\mathfrak{a}^u)^{\Omega} \right)^{1/\Omega} \right], \\ &\quad \left. \left[\left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{T}_{\mathfrak{c}} (\mathfrak{v}^l)^{\Omega} \right)^{1/\Omega}, \left(\sum_{\mathfrak{c}=1}^{\sigma} \mathfrak{T}_{\mathfrak{c}} (\mathfrak{v}^u)^{\Omega} \right)^{1/\Omega} \right] \right) \end{aligned} \quad (13)$$

where $\mathfrak{T}_{\mathfrak{c}} = \frac{\varepsilon_{\mathfrak{c}}(1 + T_{\lambda}(\mathbb{E}_{\mathfrak{c}}))}{\sum_{\mathfrak{y}=1}^{\sigma} \varepsilon_{\mathfrak{y}}(1 + T_{\lambda}(\mathbb{E}_{\mathfrak{y}}))}$, ($\mathfrak{c} = 1, 2, \dots, \sigma$) and $T_{\lambda}(\mathbb{E}_{\mathfrak{c}}) = \sum_{\substack{\mathfrak{c}=1 \\ \mathfrak{c} \neq \mathfrak{w}}}^{\sigma} S(\mathbb{E}_{\mathfrak{c}}, \mathbb{E}_{\mathfrak{w}})$.

4 | IV-PFPMATHEMATICS DESCRIPTION OF THE MADM ISSUES UNDER THE FRAMEWORK OF IV-PFVs

This section provides a mathematical description of the MADM approach by utilizing IV-PFSSWA AOs where the attribute values and weights are IV-PFVs. let $\mathfrak{h}_{\mathfrak{i}} = \{\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, \dots, \mathfrak{h}_{\mathfrak{c}}\}$ ($\mathfrak{i} = 1, 2, 3, \dots, \mathfrak{c}$) be the collection of appropriate alternatives $\mathfrak{h}_{\mathfrak{i}}$, and $\hat{\mathfrak{G}}_{\mathfrak{h}} = \{\hat{\mathfrak{G}}_1, \hat{\mathfrak{G}}_2, \hat{\mathfrak{G}}_3, \dots, \hat{\mathfrak{G}}_{\mathfrak{p}}\}$ ($\mathfrak{h} = 1, 2, 3, \dots, \mathfrak{p}$) be the finite set of criteria $\hat{\mathfrak{G}}_{\mathfrak{h}}$ having weight vectors $\varepsilon_{\mathfrak{h}} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{\mathfrak{p}})^T$ ($\mathfrak{h} = 1, 2, 3, \dots, \mathfrak{p}$) such that $\varepsilon_{\mathfrak{h}} \in [0, 1]$ and $\sum_{\mathfrak{h}=1}^{\mathfrak{p}} \varepsilon_{\mathfrak{h}} = 1$. Assume that $\mathfrak{H} = (\mathfrak{G}_{\mathfrak{ih}})_{\mathfrak{c} \times \mathfrak{p}} = ([\mathfrak{u}_{\mathfrak{ih}}^l, \mathfrak{u}_{\mathfrak{ih}}^u], [\mathfrak{a}_{\mathfrak{ih}}^l, \mathfrak{a}_{\mathfrak{ih}}^u], [\mathfrak{v}_{\mathfrak{ih}}^l, \mathfrak{v}_{\mathfrak{ih}}^u])$ be the collection of IV-PFVs, where $[\mathfrak{u}_{\mathfrak{ih}}^l, \mathfrak{u}_{\mathfrak{ih}}^u], [\mathfrak{a}_{\mathfrak{ih}}^l, \mathfrak{a}_{\mathfrak{ih}}^u], [\mathfrak{v}_{\mathfrak{ih}}^l, \mathfrak{v}_{\mathfrak{ih}}^u]$ denoting the lower and upper terms of MV, AV, and NMV in which the alternative

value h_i must satisfy the attribute value $\hat{G}_{\hat{h}}$ provided by the decision makers $\mathcal{D}$, such that $\hat{u}_{i\hat{h}}^l, \hat{u}_{i\hat{h}}^u \in [0, 1]$, $\hat{a}_{i\hat{h}}^l, \hat{a}_{i\hat{h}}^u \in [0, 1]$, $\hat{v}_{i\hat{h}}^l, \hat{v}_{i\hat{h}}^u \in [0, 1]$, $0 \leq (\hat{u}_{i\hat{h}}^u(t)) + (\hat{a}_{i\hat{h}}^u(t)) + (\hat{v}_{i\hat{h}}^u(t)) \leq 1, i = 1, 2, 3, \dots, \mathcal{C}, \hat{h} = 1, 2, 3, \dots, \mathcal{P}$.

The IV-PFWA operator is applied to the MADM issues that follow to assess the best tree (alternative) for IV-PFVs described as follows.

Step 1: Initially, we determine supports $Sup_{tt}(\mathcal{E}_{i\hat{h}}, \mathcal{E}_{i\hat{\epsilon}})$ ($i = 1, 2, \dots, \mathcal{C}; \hat{h}, \hat{\epsilon} = 1, 2, \dots, \mathcal{P}$) by applying.

$$\check{S}(\mathcal{E}_{i\hat{h}}, \mathcal{E}_{i\hat{\epsilon}}) = 1 - d(\mathcal{E}_{i\hat{h}}, \mathcal{E}_{i\hat{\epsilon}}) \quad (14)$$

$$= 1 - \frac{1}{6} \left(|\hat{u}_{i\hat{h}}^l - \hat{u}_{i\hat{\epsilon}}^l| + |\hat{a}_{i\hat{h}}^l - \hat{a}_{i\hat{\epsilon}}^l| + |\hat{v}_{i\hat{h}}^l - \hat{v}_{i\hat{\epsilon}}^l| + |\hat{u}_{i\hat{h}}^u - \hat{u}_{i\hat{\epsilon}}^u| + |\hat{a}_{i\hat{h}}^u - \hat{a}_{i\hat{\epsilon}}^u| + |\hat{v}_{i\hat{h}}^u - \hat{v}_{i\hat{\epsilon}}^u| \right)$$

For $\hat{h}, \hat{\epsilon} = 1, 2, 3, \dots, \mathcal{P}$ with $\hat{h} \neq \hat{\epsilon}$.

Step 2: To determine $\mathcal{T}(\mathcal{E}_{i\hat{h}})$ using the following Eq.(14)

$$\mathcal{T}(\mathcal{E}_{i\hat{h}}) = \sum_{\substack{\hat{\epsilon}=1 \\ \hat{h} \neq \hat{\epsilon}}}^{\mathcal{P}} Sup_{tt}(\mathcal{E}_{i\hat{h}}, \mathcal{E}_{i\hat{\epsilon}}) \quad (i = 1, 2, \dots, \mathcal{C}; \hat{h}, \hat{\epsilon} = 1, 2, \dots, \mathcal{P}) \quad (15)$$

Step 3: To compute weights $\mathcal{T}_{i\hat{h}}$ ($i = 1, 2, \dots, \mathcal{C}; \hat{h} = 1, 2, \dots, \mathcal{P}$) under the framework of IV-PFVs as follows.

$$\mathcal{T}_{i\hat{h}} = \bigoplus_{\hat{h}=1}^{\mathcal{P}} \frac{\varepsilon_{\hat{h}}(1 + \mathcal{T}(\mathcal{E}_{i\hat{h}}))}{\sum_{\hat{h}=1}^{\mathcal{P}} \varepsilon_{\hat{h}}(1 + \mathcal{T}(\mathcal{E}_{i\hat{h}}))}$$

Where $\mathcal{T}_{i\hat{h}} \geq 0$, ($i = 1, 2, \dots, \mathcal{C}, \hat{h} = 1, 2, \dots, \mathcal{P}$), and $\sum_{\hat{h}=1}^{\mathcal{P}} \mathcal{T}_{i\hat{h}} = 1, i = 1, 2, 3, \dots, \mathcal{C}$.

Step 4: To compute the final accumulative value $\mathcal{E}_i$ ($i = 1, 2, \dots, \mathcal{C}$) by applying the IV-PFSSWA AOs.

$$\begin{aligned} \mathcal{E}_i &= IV - PFSSWA(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \dots \mathcal{E}_{\mathcal{P}}) = \bigoplus_{\hat{h}=1}^{\mathcal{P}} \frac{\varepsilon_{\hat{h}}(1 + \mathcal{T}(\mathcal{E}_{i\hat{h}}))}{\sum_{\hat{h}=1}^{\mathcal{P}} \varepsilon_{\hat{h}}(1 + \mathcal{T}(\mathcal{E}_{i\hat{h}}))} \\ &= \left(\left[1 - \left(\sum_{\hat{h}=1}^{\mathcal{P}} \varepsilon_{\hat{h}} (1 - \hat{u}_{i\hat{h}}^l)^{\omega} \right)^{1/\omega}, 1 - \left(\sum_{\hat{h}=1}^{\mathcal{P}} \varepsilon_{\hat{h}} (1 - \hat{u}_{i\hat{h}}^u)^{\omega} \right)^{1/\omega} \right], \right. \\ &\quad \left[\left(\sum_{\hat{h}=1}^{\mathcal{P}} \varepsilon_{\hat{h}} (\hat{a}_{i\hat{h}}^l)^{\omega} \right)^{1/\omega}, \left(\sum_{\hat{h}=1}^{\mathcal{P}} \varepsilon_{\hat{h}} (\hat{a}_{i\hat{h}}^u)^{\omega} \right)^{1/\omega} \right] \\ &\quad \left. \left[\left(\sum_{\hat{h}=1}^{\mathcal{P}} \varepsilon_{\hat{h}} (\hat{v}_{i\hat{h}}^l)^{\omega} \right)^{1/\omega}, \left(\sum_{\hat{h}=1}^{\mathcal{P}} \varepsilon_{\hat{h}} (\hat{v}_{i\hat{h}}^u)^{\omega} \right)^{1/\omega} \right] \right) \end{aligned}$$

Step 5: To compute the score values $\delta(\mathcal{E}_i)$ ($i = 1, 2, \dots, \mathcal{C}$) of the entire IV-PFVs $\mathcal{E}_i$ ($i = 1, 2, \dots, \mathcal{C}$).

Step 6: Choose the most advantageous optimal results by listing the ranking results of alternatives h_i ($i = 1, 2, 3, \dots, \mathcal{C}$) in an appropriate sequence.

Step 7: End

4.1 | NUMERICAL EXAMPLE

Tree plantations play an important role as they enhance human lives and provide necessities. The trees decompose food materials and absorb carbon dioxide (CD) during the process of photosynthesis. They are helping to purify the atmosphere by absorbing CD and releasing oxygen. They provide shade to houses, roads, and streets. A tree plantation ensures a never-ending source of oxygen. Let we have to plant four different types of trees (alternatives) $h_i (i = 1, 2, 3, 4)$ that protect the environment and also enhance human lives. The following characteristics of tree plantation $\hat{G}_h (h = 1, 2, 3, 4)$ are helpful for tree plantation which is described as

$\hat{G}_1$ = Trees provide shade and reduce wind speed.

$\hat{G}_2$ = Trees provide habitat for birds, mammals, and insects.

$\hat{G}_3$ = Trees enhance the beauty of the environment.

$\hat{G}_4$ = Trees control noise pollution and clean the air.

The decision-maker must examine the alternatives $h_i (i = 1, 2, 3, 4)$ based on the IV-PFVs for the above-mentioned characteristics whose weight vectors are $\varepsilon_{\mathfrak{e}} = (0.23, 0.35, 0.10, 0.32)$, and the resulting matrix based on the IV-PFVs $\mathfrak{N} = (\mathfrak{G}_{ih})_{\mathfrak{e} \times \mathfrak{p}} = ([\hat{u}_{ih}^l, \hat{u}_{ih}^u], [\hat{a}_{ih}^l, \hat{a}_{ih}^u], [\hat{v}_{ih}^l, \hat{v}_{ih}^u])$ are displayed in Table 1.

Table 1: The IV-PF Decision matrix $\mathfrak{N}$

	h_1	h_2	h_3	h_4
$\hat{G}_1$	$\begin{pmatrix} [0.04, 0.050], \\ [0.214, 0.256], \\ [0.237, 0.327] \end{pmatrix}$	$\begin{pmatrix} [0.020, 0.123], \\ [0.242, 0.268], \\ [0.474, 0.481] \end{pmatrix}$	$\begin{pmatrix} [0.050, 0.172], \\ [0.213, 0.293], \\ [0.492, 0.513] \end{pmatrix}$	$\begin{pmatrix} [0.020, 0.112], \\ [0.128, 0.231], \\ [0.418, 0.569] \end{pmatrix}$
$\hat{G}_2$	$\begin{pmatrix} [0.124, 0.231], \\ [0.050, 0.144], \\ [0.376, 0.423] \end{pmatrix}$	$\begin{pmatrix} [0.186, 0.245], \\ [0.070, 0.132], \\ [0.385, 0.476] \end{pmatrix}$	$\begin{pmatrix} [0.147, 0.275], \\ [0.151, 0.231], \\ [0.312, 0.432] \end{pmatrix}$	$\begin{pmatrix} [0.169, 0.289], \\ [0.090, 0.156], \\ [0.342, 0.492] \end{pmatrix}$
$\hat{G}_3$	$\begin{pmatrix} [0.132, 0.142], \\ [0.145, 0.175], \\ [0.030, 0.425] \end{pmatrix}$	$\begin{pmatrix} [0.173, 0.183], \\ [0.246, 0.235], \\ [0.050, 0.423] \end{pmatrix}$	$\begin{pmatrix} [0.214, 0.225], \\ [0.342, 0.352], \\ [0.070, 0.337] \end{pmatrix}$	$\begin{pmatrix} [0.214, 0.246], \\ [0.128, 0.321], \\ [0.09, 0.187] \end{pmatrix}$
$\hat{G}_4$	$\begin{pmatrix} [0.134, 0.236], \\ [0.060, 0.155], \\ [0.162, 0.193] \end{pmatrix}$	$\begin{pmatrix} [0.195, 0.252], \\ [0.080, 0.138], \\ [0.200, 0.230] \end{pmatrix}$	$\begin{pmatrix} [0.020, 0.162], \\ [0.193, 0.245], \\ [0.120, 0.172] \end{pmatrix}$	$\begin{pmatrix} [0.186, 0.292], \\ [0.112, 0.192], \\ [0.120, 0.276] \end{pmatrix}$

4.2 | THE MADM STRATEGY BY APPLYING IV-PFSSWPA OPERATOR

The MADM strategy with IV-PFVs is solved by using the IV-PFSSWPA operator which brings the following computations to a successful conclusion.

Step 1: First of all, we determine supports $Suptt(\mathcal{E}_{ih}, \mathcal{E}_{i\mathfrak{e}}) (i = 1, 2, \dots, \mathfrak{e}; h = 1, 2, \dots, \mathfrak{p})$ by applying $\check{S}(\mathcal{E}_{ih}, \mathcal{E}_{i\mathfrak{e}}) = 1 - d(\mathcal{E}_{ih}, \mathcal{E}_{i\mathfrak{e}})$, then we have

$$\left(\begin{array}{l} \check{S}_{11,12} = 0.294, \check{S}_{11,13} = 0.438, \check{S}_{11,14} = 0.334, \check{S}_{12,13} = 0.436, \check{S}_{12,14} = 0.528, \check{S}_{13,14} = 0.437, \\ \check{S}_{21,22} = 0.448, \check{S}_{21,23} = 0.396, \check{S}_{21,24} = 0.025, \check{S}_{22,23} = 0.269, \check{S}_{22,24} = 0.545, \check{S}_{23,24} = 0.321, \\ \check{S}_{31,32} = 0.496, \check{S}_{31,33} = 0.134, \check{S}_{31,34} = 0.204, \check{S}_{32,33} = 0.290, \check{S}_{32,34} = 0.358, \check{S}_{33,34} = 0.434, \\ \check{S}_{41,42} = 0.532, \check{S}_{41,43} = 0.034, \check{S}_{41,44} = 0.146, \check{S}_{42,43} = 0.190, \check{S}_{42,44} = 0.498, \check{S}_{43,44} = 0.685, \end{array} \right)$$

Step 2: To determine $T(E_{ih})$ by applying $T(E_{ih}) = \sum_{\substack{\mathfrak{E}=1 \\ \mathfrak{h} \neq \mathfrak{E}}}^{\mathfrak{P}} \text{Sup}tt(E_{ih}, E_{i\mathfrak{E}})$ ($i = 1, 2, \dots, \mathfrak{C}; \mathfrak{h}, \mathfrak{E} = 1, 2, \dots, \mathfrak{P}$), then we obtain

$$\left(\begin{array}{l} T_{11} = 1.066, T_{12} = 1.258, T_{13} = 1.350, T_{14} = 1.299, \\ T_{21} = 0.869, T_{22} = 1.262, T_{23} = 0.986, T_{24} = 0.891, \\ T_{31} = 0.834, T_{32} = 1.144, T_{33} = 0.857, T_{34} = 0.996, \\ T_{41} = 0.712, T_{42} = 1.220, T_{43} = 0.909, T_{44} = 1.330, \end{array} \right)$$

Step 3: Now we calculate the weights $\mathfrak{T}_{ih}$ ($i = 1, 2, \dots, \mathfrak{C}; \mathfrak{h} = 1, 2, \dots, \mathfrak{P}$) by applying $\mathfrak{T}_{ih} = \frac{\mathfrak{P}}{\mathfrak{h}=1} \frac{\varepsilon_{\mathfrak{h}}(1+T(E_{ih}))}{\sum_{\mathfrak{h}=1}^{\mathfrak{P}} \varepsilon_{\mathfrak{h}}(1+T(E_{ih}))}$ corresponding to the given criteria $\hat{G}_{\mathfrak{h}}$ ($\mathfrak{h} = 1, 2, 3, \dots, \mathfrak{P}$) under the framework of IV-PFVs as follows

$$\left(\begin{array}{l} \mathfrak{T}_{11} = 0.213, \mathfrak{T}_{12} = 0.354, \mathfrak{T}_{13} = 0.103, \mathfrak{T}_{14} = 0.330, \\ \mathfrak{T}_{21} = 0.212, \mathfrak{T}_{22} = 0.391, \mathfrak{T}_{23} = 0.098, \mathfrak{T}_{24} = 0.299, \\ \mathfrak{T}_{31} = 0.211, \mathfrak{T}_{32} = 0.376, \mathfrak{T}_{33} = 0.093, \mathfrak{T}_{34} = 0.320, \\ \mathfrak{T}_{41} = 0.187, \mathfrak{T}_{42} = 0.369, \mathfrak{T}_{43} = 0.091, \mathfrak{T}_{44} = 0.354, \end{array} \right)$$

Step 4: To find the final accumulative value E_i ($i = 1, 2, \dots, \mathfrak{C}$) by applying the IV-PFSSWA AOs, and considering the value of parameter $\Omega = -1$, then we have.

Table 2: Aggregated result of tree plantation by the IV-PFSSPWA operator

h_1	h_2	h_3	h_4
$\left(\begin{array}{l} [0.1117, 0.1913], \\ [0.1348, 0.1664], \\ [0.1365, 0.2908] \end{array} \right)$	$\left(\begin{array}{l} [0.1572, 0.2183], \\ [0.1870, 0.1578], \\ [0.2033, 0.3580] \end{array} \right)$	$\left(\begin{array}{l} [0.0973, 0.2159], \\ [0.1615, 0.2552], \\ [0.1777, 0.2926] \end{array} \right)$	$\left(\begin{array}{l} [0.1556, 0.2587], \\ [0.1540, 0.1888], \\ [0.1825, 0.3516] \end{array} \right)$

Step 5: To compute the score values $\delta(E_i)$ ($i = 1, 2, \dots, \mathfrak{C}$) for the appropriate selection of trees (alternative) that are helpful to protect the environment and enhance human lives.

Table 3: The score values of tree plantation by the IV-PFSSPWA operator

$\delta(h_1)$	$\delta(h_2)$	$\delta(h_3)$	$\delta(h_4)$
0.1160	0.1311	0.0968	0.1429

Step 6: The best preferable result is h_4 .

The following Figure 1 displays the score values of alternatives under the framework of the IV-PFSSPWA operator.

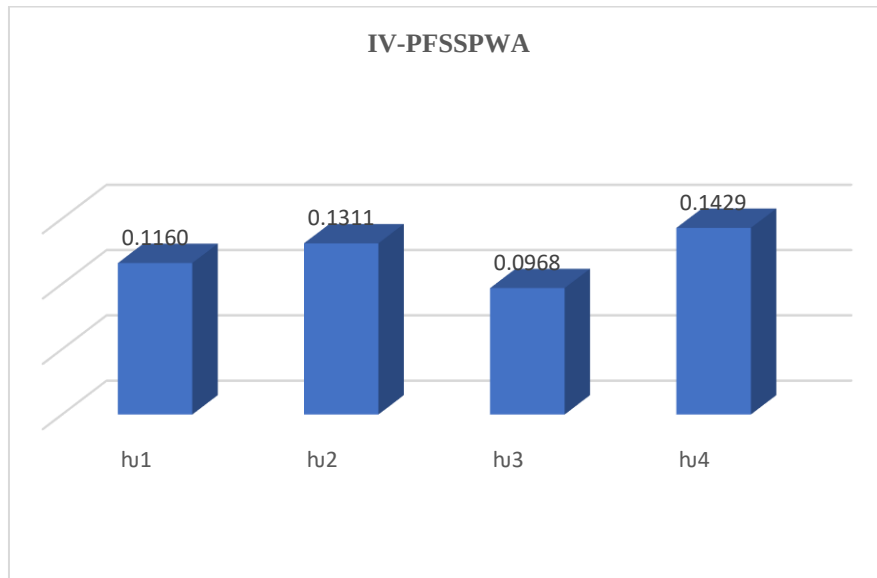


Figure 1: portrays the ranking results of the IV-PFSSPWA operator.

4.3 | INFLUENCE OF PARAMETER Ω

We see the impact of characterized parameter Ω on the ranking and ordering outcomes of score values is evaluated by applying the SSWA AOs under the framework of IV-PFVs. The following Table 4 shows the ranking and ordering results of alternatives by using different parametric values, when we take the values of parameter $\Omega \geq -20$ the ranking list of alternatives are $h_2 > h_4 > h_1 > h_3$. If we take the values of parameter $\Omega = -25$ to $\Omega = -70$ the above ranking list of alternative changes to $h_2 > h_4 > h_3 > h_1$. The ranking list $h_2 > h_4 > h_3 > h_1$ always remains the same when we take the parametric values $\Omega \leq -70$.

Table 4: The ranking and ordering of alternatives by the IV-PFSSPWA operator

Ω	Ordering of Score (Sc) values	Ranking results
$\Omega = -5$	$Sc(h_2) > Sc(h_4) > Sc(h_1) > Sc(h_3)$	$h_2 > h_4 > h_1 > h_3$
$\Omega = -9$	$Sc(h_2) > Sc(h_4) > Sc(h_1) > Sc(h_3)$	$h_2 > h_4 > h_1 > h_3$
$\Omega = -15$	$Sc(h_2) > Sc(h_4) > Sc(h_1) > Sc(h_3)$	$h_2 > h_4 > h_1 > h_3$
$\Omega = -20$	$Sc(h_2) > Sc(h_4) > Sc(h_1) > Sc(h_3)$	$h_2 > h_4 > h_1 > h_3$
$\Omega = -25$	$Sc(h_2) > Sc(h_4) > Sc(h_3) > Sc(h_1)$	$h_2 > h_4 > h_3 > h_1$
$\Omega = -35$	$Sc(h_2) > Sc(h_4) > Sc(h_3) > Sc(h_1)$	$h_2 > h_4 > h_3 > h_1$
$\Omega = -45$	$Sc(h_2) > Sc(h_4) > Sc(h_3) > Sc(h_1)$	$h_2 > h_4 > h_3 > h_1$
$\Omega = -57$	$Sc(h_2) > Sc(h_4) > Sc(h_3) > Sc(h_1)$	$h_2 > h_4 > h_3 > h_1$
$\Omega = -60$	$Sc(h_2) > Sc(h_4) > Sc(h_3) > Sc(h_1)$	$h_2 > h_4 > h_3 > h_1$
$\Omega = -70$	$Sc(h_2) > Sc(h_4) > Sc(h_3) > Sc(h_1)$	$h_2 > h_4 > h_3 > h_1$

The following Figure 2 displays the ranking and ordering results of the best tree (alternatives) by using different parametric values $\Omega = -5$ to $\Omega = -70$.

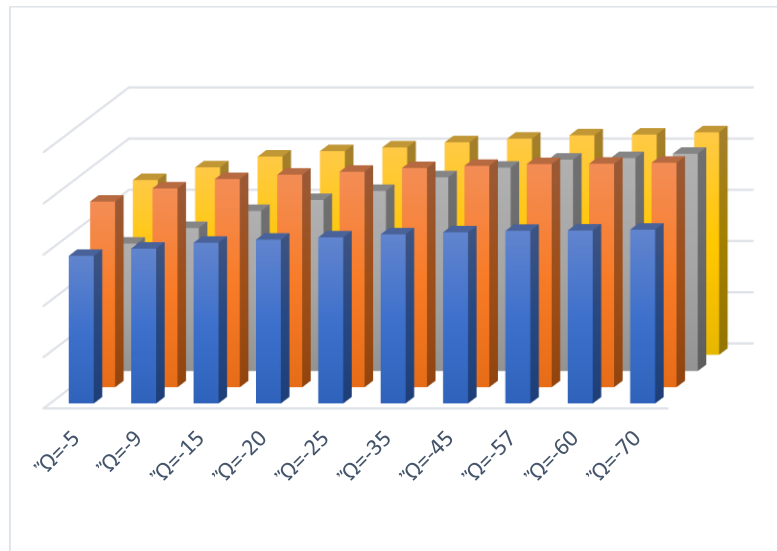


Figure 2 : portrays the ranking results of alternatives

5 | COMPARATIVE ANALYSIS

To determine the beneficial effects of the recommended approach and also show their applicability, we compare the novel invented methodology with those of the previously existing methodologies applied under the framework of IV-PFVs, IV-IFVs, and Complex IV-IFVs (CIV-IFVs). The final ranking outcomes of the novel invented strategies as IV-PFSSPA and IV-PFSSPWA operators with previously discussed methodologies including IV-PF weighted average (IV-PFWA) operator, IV-PF weighted geometric (IV-PFWG) operator initially developed by Kahraman [43], IV-PF Aczel-Alsina WA (IV-PFA-AWA) operator, IV-PF Aczel-Alsina WG (IV-PFAAWG) operator established by Jabeen et al. [32], CIV-IFWA operator, CIV-IFWG operator originally developed by Garg and Rani [44], IV-IF Hamacher WA (IV-IFHWA) operator, and IV-IF Hamacher WG (IV-IFHWG) operator presented by Liu [45]. These above-mentioned AOs are portrayed in the following Table 4 and the comprehensive ranking outcomes of our proposed AOs with the previously defined AOs are the same as $h_4 > h_2 > h_1 > h_3$. It means that h_4 is the best optimal choice. Some of these AOs including CIV-IFWA, CIV-IFWG, IV-IFHWA, and IV-IFHWG operators are not capable of dealing with MADM difficulties under the framework of IV-PFVs. In such scenarios, based on the information provided above our proposed AOs are more dominant, efficient, and functioning properly because the PA operator may ameliorate some of the adverse consequences of falsified information provided by the detrimental decision-makers.

Table 5: shows the comparison results.

AOs	Ranking list
IV-PFSSPA (Current work)	$h_4 > h_2 > h_1 > h_3$
IV-PFSSPWA (Current work)	$h_4 > h_2 > h_1 > h_3$
IV-PFWA [43]	$h_4 > h_2 > h_1 > h_3$
IV-PFWG [43]	$h_4 > h_2 > h_1 > h_3$
IV-PFAAWA [32]	$h_4 > h_2 > h_1 > h_3$
IV-PFAAWG [32]	$h_4 > h_2 > h_1 > h_3$
CIV-IFWA [44]	Not possible to apply
CIV-IFWG [44]	Not possible to apply

IV-IFHWA [45]	Not possible to apply
IV-IFHWG [45]	Not possible to apply

Figure 3 portrayed the comparison ranking results of alternatives by using our recommended AOs including the IV-PFSSPA operator and IV-PFPFSSPWA operator with previously defined AOs as follows.

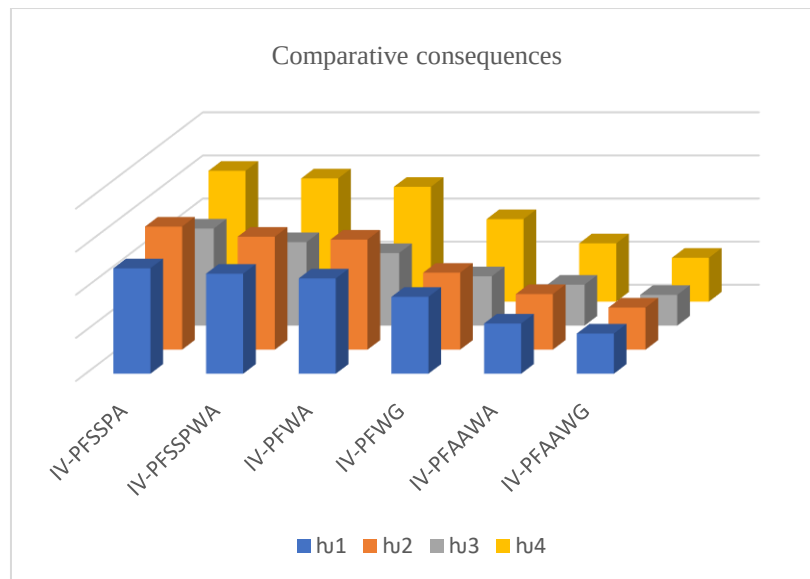


Figure 3: shows the comparative results

5 | CONCLUSION

The IV-PFS is an innovative technique for resolving MADM issues that has multiple applications in various research disciplines and other systematic approaches. In this paper, we have discussed some PAOs that are beneficial in minimizing some of the adverse reactions of deceptive data, and the SS t-norm and SS t-conorm are used to establish some novel AOs such as SSPA and SSPWA operators for IV-PFVs. Some important features and numerical demonstrations of these AOs are comprehensively discussed. Additionally, we implemented some of the suggested AOs to establish an approach for resolving MADM issues by applying IV-PF data. Finally, we provided an example to explain the importance and usefulness of the tree selection scenario which is helpful to protect the environment and also enhance human lives. By using the specified numerical example, we have approximately demonstrated the ranking results of alternatives by fluctuating the parameter η under the environment of the IV-PFSSWPA operator.

In future research, we will concentrate on expanding and applying to a wider range of disciplines and ambiguous scenarios such as complex T-SFSs (CT-SFSs) [46], CT-S Dual Hesitant FSs (CT-SDHFSs) [47], and Cq-R Dombi FSs (Cq-RODFSs) [48].

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