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Selection and Assessment of Enterprise Resource Planning System by Using Bipolar Complex Fuzzy MCDM Approach Based on Power Muirhead Mean Operators

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ABSTRACT

The selection of enterprise resource planning (ERP) systems is one of the most strategic and profound decisions for organizations since it impacts their performance and competitive position. Due to the complexity and variety of the ERP systems that have evolved over the years, the decision-making (DM) process has become more difficult. These complexities have raised several major research gaps in the literature and the selection methodologies of ERP systems. One significant weakness is that the current strategies pay attention only to the positive aspects of the attributes of ERP systems and ignore the negative aspects of the attributes of ERP systems. As a result, the decision maker may end up with an ERP evaluation that is either unbalanced or skewed, and therefore not entirely accurate. Besides, most of the existing methods do not consider extra fuzzy information concerning the attributes of ERP, which leads to the loss of some important information needed for making a wise decision. Thus, in this manuscript, we propose a new MADM method, namely bipolar complex fuzzy MADM (BCF-MADM), which not only covers positive and negative aspects of ERP attributes but also includes additional fuzzy information and reduces the influence of the extreme values of the decision-makers' opinions. To overcome the problem of outlier assessments, the power average (PA) operator is used, and the Muirhead mean (MM) operator is used to consider the interrelation between attribute arguments, which provides more flexibility in the aggregation process, and new power MM operators and power dual MM operators are proposed for bipolar complex fuzzy set (BCFS). These operators are used in the aggregation process of the developed MADM problem. After that, we analyze a case study related to the selection of an ERP system and then compare the anticipated theory with certain existing ones to demonstrate the supremacy of the anticipated theory.

Keywords: Enterprise resource planning; power Muirhead mean operator; bipolar complex fuzzy set; Decision Making.

(MSC2020) Codes: 03E72, 03E75, 90B50, 90C70

1. | INTRODUCTION

ERP is an acronym for enterprise resource planning; what is ERP? The simplest way to define ERP is to think about all the core business processes needed to run a company, such as finance, human resources, production, purchasing and supply, services, and others. In its simplest form, ERP is the solution that enables the most efficient handling of all these processes in an integrated manner. It is commonly known as the system of record of the organization. However, the ERP systems of today are far from simple and bear very little similarity to the original ERP system that was developed many years ago. They are now delivered through the cloud and leverage new technologies, including AI and machine learning, to offer intelligent automation, higher productivity, and real-time business intelligence. It also integrates business with internal and external business partners and networks across the globe, which provides today's competitive environment: collaboration, agility, and speed.

Also referred to as the heart of an enterprise, ERP software offers the automation, integration, and intelligence that is required in the execution of all enterprise operations. Ideally, most or all of an organization's data should be in the ERP system in order to ensure that the business has a centralized source for data. Finance needs an ERP to close books in a short time. Sales requires ERP to handle all the orders from customers. Logistics in its true sense requires the efficient functioning of ERP software to deliver the right product and service to the customers at the right time. Accounts payable requires ERP to pay the suppliers the right amount and at the right time. Management requires real-time information with regard to the company's performance in order to make appropriate decisions at the right time. Banks and shareholders also need to have precise records of the company's financial status, and thus they rely on the data and analysis provided by the ERP system. This is an indication that ERP software is very crucial to businesses as more and more firms implement it. G2 says, "The global ERP Software market is expected to grow at a CAGR of 10. 2% during the forecast period of 2019-2026, reaching US$78. 40 billion. The importance, emergence, and issues in ERP systems were devised by AlBar et al. [1], and its strategic importance was analyzed by Herath and Wijenavake [2].

The benefits that a good ERP system provides can be diverse — depending on the implementation strategy. For instance, the advantages of cloud ERP are unlike those of an on-premise ERP system. The major benefits of ERP are shown in Figure 1.

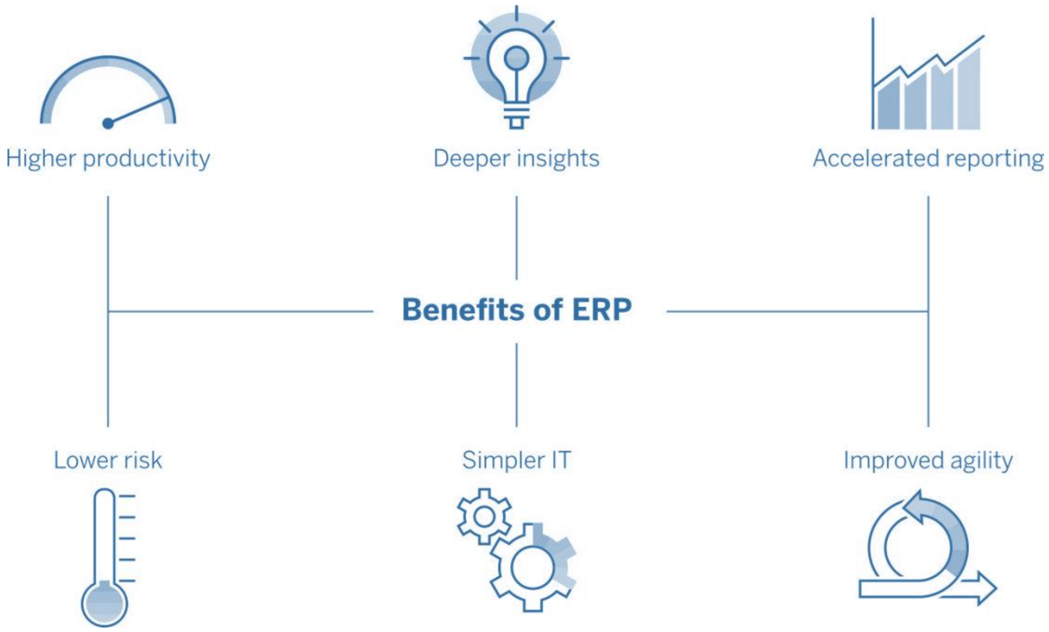

Figure 1. The benefits of ERP.

An ERP system can also be referred to as an ERP suite and comprises interrelated modules or business applications that are connected and use the same database. Individual modules of ERP are generally specific to a single business function, although they are integrated and share the same database to address the company's requirements. Finance and accounting, human resources, sales and marketing, procurement, logistics, and supply chain are some of the frequent choices. It is a la carte, and firms select the module they need and can then add on and expand from there. ERP systems also meet industry needs, either as a built-in module or incorporated add-ons that operate in parallel with the ERP applications. ERP software can be purchased through a Cloud subscription model, which is a software as a service model, or through a license, which is an on-premises model.

ERP systems encompass a range of modules of different types. Every ERP module is responsible for particular business processes – for example, financial ones, purchasing ones, or manufacturing ones – and gives the workers in the particular department all the transactions and the required data they need for their work. Each module integrates with the ERP system, which provides real-time, consistent data that is available to all departments. The basic modules of ERP are shown in Figure 2.

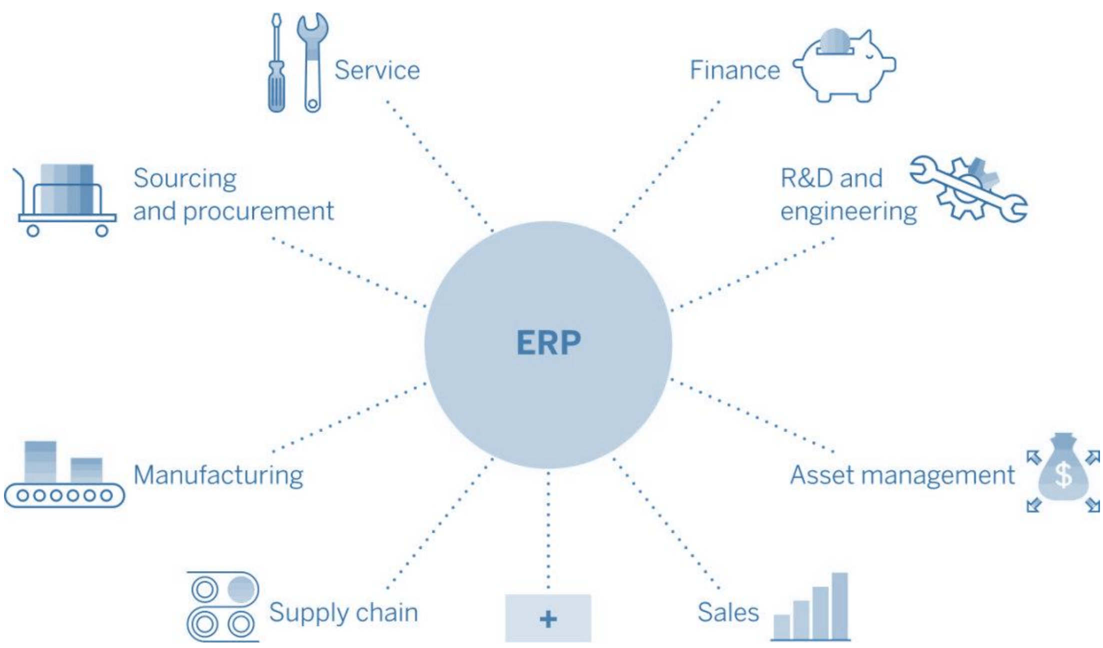

Figure 2. The modules of ERP.

The history of ERP is demonstrated in Figure 3.

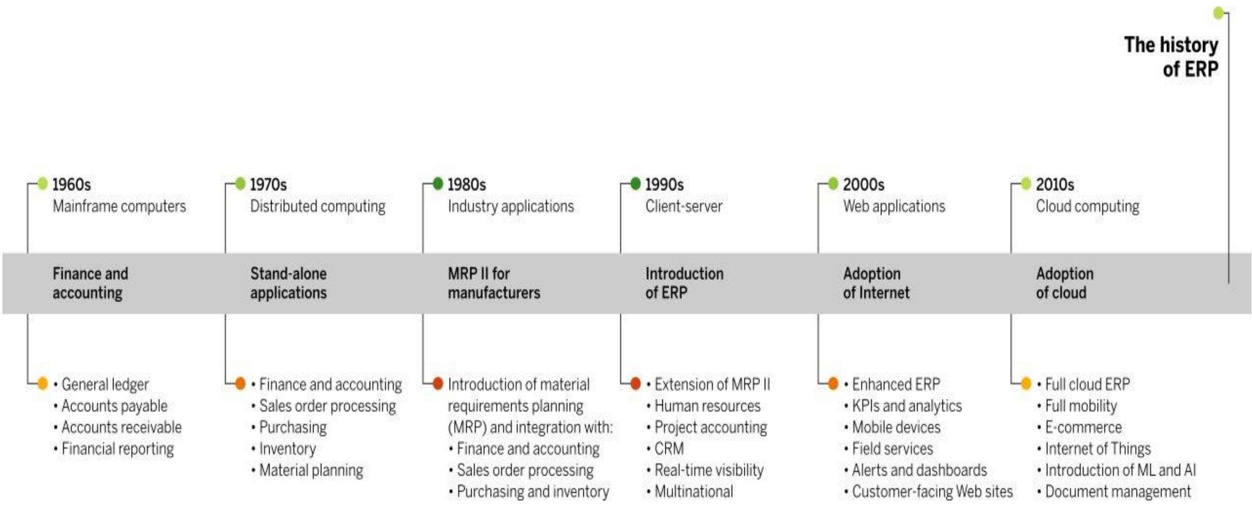

Figure 3. The history of ERP.

The choice of ERP systems is regarded as the MADM problem because it is a process of selecting one of the multiple options under consideration, taking into account several attributes that may be conflicting. There are factors such as the cost of the software, functionality, scalability, reputation of the vendor, time taken to implement the software, technical support available, and compatibility with the existing systems in the organization. All of these criteria can be of varying significance depending on the specifics of the organization in question. Also, the decision-making (DM) process involves not only the numerical data but also the qualitative aspects, which makes it rather intricate. The MADM approach offers a clear guideline on how to address these various criteria, how to assign weights to them, and how to arrive at a decision that is most appropriate within the context of the organization's strategic direction and operational needs. UȚĂ et al. [3] emphasized the criteria for the selection of ERP software. Their work was also informative in that they stressed the need for a clear framework for assessing ERP systems. They may have debated the issues that an organization should take into consideration when selecting the ERP system, including functionality, cost, scalability, and the reputation of the vendor. This research helped to identify that ERP selection is a complex process and requires evaluation criteria that cover all aspects of the issue. After that, Molnar et al. [4] extended the research on the ERP selection process. It is possible that their study looked at the different phases that organizations go through to select an ERP system, from identification of needs to decision making. Demydenko [5] advanced the research by developing multi-criteria optimization models for the selection of ERP systems. This approach is a more complex way of DM, most probably involving the use of equations to weigh a number of, sometimes, mutually exclusive factors.

1.1. | Fuzzy Set and its Hybrid Structures

Zadeh [6] originated the idea of the fuzzy set (FS) because, in numerous tricky situations, it is almost impossible for experts or decision-makers to render their values in the form of a crisp number. The FS manages the uncertainties in the data, which employs the grade of satisfaction to describe the knowledge. The theory of FS has attracted a lot of attention from many scholars all over the world, who have studied the technical and conceptual properties of FS. The recent investigation on FS and its application includes supply chain [7, 8], genetic algorithms [9, 10], and business and economics [11-13]. Further, Xu et al. [14] analyzed a fuzzy MADM approach for the selection of an ERP system. Nikdel et al. [15] devised the fuzzy k-nearest neighbors' algorithm. Cui et al. [16] studied fuzzy rare itemset mining. Chen and Lu [17] studied correlation analysis in the fuzzy environment, and Jiang et al. [18] established fuzzy entropy. The description of real-valued bipolar is not able to describe the fuzziness and uncertainties like unipolar crisp values were not able to do. Therefore, the origination of bipolar FS (BFS) was necessary for dealing with both fuzziness and polarity, and Zhang [19] originated it in 1994. The BFS manages uncertainty data involving both positive and negative poles. The positive pole is described by the positive grade of satisfaction that is placed in $[0,1]$ and the negative pole is described by the negative grade of satisfaction that is placed in $[-1,0]$. Deva and Filex [20] originated the DEMATEL procedure, Riaz and Tehrim [21] explored the VIKOR procedure, Riaz et al. [22] gave the SIR procedure, and Jana [23] described MABAC; Akram et al. [24] devised TOPSIS and ELECTRE-I, and Shumaiza et al. [25] interpreted ELECTRE-II for bipolar fuzzy (BF) information. Some aggregation operators (AOs) for BF information were originated by Jana et al. [26, 27] and Wei et al. [28]. Various graphs and its application are discussed in [29-32]. The BF system of linear equations was devised by Akram et al. [33], and discussed this concept with a polynomial parametric form was devised by Akram et al. [34].

The FS manages the uncertainties in one dimension and can't manage the 2nd dimension; thus, Ramot et al. [35] originated the idea of complex FS (CFS). The CFS manages the two-dimensional data involving uncertainties and ambiguities. There is a grade of satisfaction in the CFS for describing the knowledge of an element in the polar model and placing it in the unit disc of a complex plane. Tamir et al. [36] originated another model of the grade of satisfaction, which is a Cartesian model that places elements in the unit square of the complex plane. Bi et al. [37] originated complex fuzzy (CF) arithmetic AOs, Bi et al. [38] explored CF geometric AOs, and Hu et al. [39] investigated CF power AOs in the setting of CFS. To model the positive and negative poles and the 2nd dimension in a single structure, we require a more modified and advanced structure, and that is BCFS originated by Mahmood and Rehman [40]. This structure generalized all the above-discussed theories. There is a positive grade of satisfaction in the BCFS for describing the positive knowledge of an element that is placed in $[0,1]+\iota[0,1]$ and also there is a negative grade of satisfaction in the BCFS describing the negative knowledge of an element that is placed in $[-1,0]+\iota[-1,0]$. Mahmood and Rehman [41] originated Dombi, Mahmood et al. [42] diagnosed Hamacher, Mahmood et al. [43] established Bonferroni mean (BM), Mahmood et al. [44] originated Heronian mean (HM), Ur Rehman et al. [45] presented Frank, Mahmood and Ur Rehman [46] described Maclaurin symmetric mean (MSM) AOs in the environment of BCFS.

1.2. | Motivation and Contribution

The complexities are increasing in real-life dilemmas over time, and with that, there are creating research gaps and questions in the existing literature. Some of the research gaps are identified as follows.

- Some of the attributes of the ERP can have both positive and negative aspects, and in the existing literature, merely the positive aspects of the ERP are considered, while negative aspects are ignored.
- The extra fuzzy information related to the attributes of the ERP is also ignored in the prevailing literature on ERP.
- There are a few circumstances where the experts or decision makers present very low or very high values of attributes, which have a bad effect on the final ranking outcomes. In this case, the power average (PA) operator originated by Yager [47] is a very useful AO that allows the assessed values to be commonly upheld and improved. Hence, we employ the PA operator to mitigate that bad effect by defining numerous weights generated with the assistance of support measures.
- In a few real-life DM dilemmas, the argument of attribute depend on each other and are not independent. Thus, the interrelatedness between the attributes should be considered. The MSM operators [48], BM operators [49], and HM operators [50] capture the interrelatedness among the arguments of attributes. But the Muirhead mean (MM) [51] is more advanced and beneficial than these operators, and the advantages of the MM operators are investigated by Liu and Li [52] and

Lui and You [53]. Additionally, certain prevailing AOs are the particular cases of MM operators like BM [49] and MSM operators [48]. There is a parameter vector in the MM operator that enhances the flexibility in the aggregating procedure.

- The advantages and benefits of PA, MM, and dual MM operators at the same time within BCFS are missing.

To overcome these gaps, in this script, we first introduced new AOs by combining PA and MM, as well as PA and dual MM in the structure of BCFS. The names of these operators are BCF power Muirhead mean (BCFPMM), BCF weighted power Muirhead mean (BCFWPMM), BCF power dual Muirhead mean (BCFPDMM), and BCF weighted power dual Muirhead mean (BCFWPDMM) operators. After that, we devise a MADM method within BCFS for tackling the ERP selection problem and other MADM dilemmas, where bipolarity and extra fuzzy information are involved. This method will be known as the BCF-MADM method. The deduced MADM approach within BCFS is based on the deduced operators, which will help to aggregate the assessment values of the ERP. Thus, the combination of BCFS and devised operators will give us a more robust approach for solving MADM problems.

The remaining paper is organized as: Section 2 discusses the basics of BCFS, PA, and MM operators. In Section 3, we introduce BCFPMM, BCFWPMM, BCFPDMM, and BCFWPDMM operators and their properties. Section 4 includes the BCF-MADM method and the related case study of ERP systems. Section 5 discusses the comparative analysis, and Section 6 discusses the conclusion of the paper.

2. | PRELIMINARIES

Here, we recall the notion of BCFS, related operations, MM, and dual MM operators.

Definition 1: [40] Consider a fixed set $\mathfrak{E}$, then BCFS over $\mathfrak{E}$ originated as

$$\mathbb{U}_{BCFS} = \left\{\left(\mathbb{e}, \left(\mathfrak{J}^{P}_{\mathbb{U}_{BCFS}}(\mathbb{e}), \mathfrak{J}^{N}_{\mathbb{U}_{BCFS}}(\mathbb{e})\right)\right) \mid \mathbb{e} \in \mathfrak{E}\right\}$$
$$= \left\{\left(\mathbb{e}, \left(\mathfrak{J}^{RP}_{\mathbb{U}_{BCFS}}(\mathbb{e}) + \iota\, \mathfrak{J}^{IP}_{\mathbb{U}_{BCFS}}(\mathbb{e}), \mathfrak{J}^{RN}_{\mathbb{U}_{BCFS}}(\mathbb{e}) + \iota\, \mathfrak{J}^{IN}_{\mathbb{U}_{BCFS}}(\mathbb{e})\right)\right) \mid \mathbb{e} \in \mathfrak{E}\right\} \quad (1)$$

The positive grade of satisfaction would be indicated by $\mathfrak{J}^{P}_{\mathbb{U}_{BCFS}}(\mathbb{e})$ and the negative grade of satisfaction would be indicated $\mathfrak{J}^{N}_{\mathbb{U}_{BCFS}}(\mathbb{e})$ and $\mathfrak{J}^{RP}_{\mathbb{U}_{BCFS}}(\mathbb{e}), \mathfrak{J}^{IP}_{\mathbb{U}_{BCFS}}(\mathbb{e}) \in [0,1]$, $\mathfrak{J}^{RN}_{\mathbb{U}_{BCFS}}(\mathbb{e}), \mathfrak{J}^{IN}_{\mathbb{U}_{BCFS}}(\mathbb{e}) \in [-1,0]$.

Remark 1: The set $\mathbb{U}_{BCFS} = \left(\mathfrak{J}^{P}_{\mathbb{U}_{BCFS}}, \mathfrak{J}^{N}_{\mathbb{U}_{BCFS}}\right) = \left(\mathfrak{J}^{RP}_{\mathbb{U}_{BCFS}} + \iota\, \mathfrak{J}^{IP}_{\mathbb{U}_{BCFS}}, \mathfrak{J}^{RN}_{\mathbb{U}_{BCFS}} + \iota\, \mathfrak{J}^{IN}_{\mathbb{U}_{BCFS}}\right)$ would be indicated by the BCF number (BCFN)

Definition 2: [41] Consider $\mathbb{U}_{BCFS} = \left(\mathfrak{J}^{P}_{\mathbb{U}_{BCFS}}, \mathfrak{J}^{N}_{\mathbb{U}_{BCFS}}\right) = \left(\mathfrak{J}^{RP}_{\mathbb{U}_{BCFS}} + \iota\, \mathfrak{J}^{IP}_{\mathbb{U}_{BCFS}}, \mathfrak{J}^{RN}_{\mathbb{U}_{BCFS}} + \iota\, \mathfrak{J}^{IN}_{\mathbb{U}_{BCFS}}\right)$ is a BCFN, then Eq. (2) provides the score value and Eq. (3) provides the accuracy of a BCFN $\mathbb{U}_{BCFS}$.

$$\mathcal{S}(\mathbb{U}_{BCFS}) = \frac{1}{4}\left(2 + \mathfrak{J}^{RP}_{\mathbb{U}_{BCFS}} + \mathfrak{J}^{IP}_{\mathbb{U}_{BCFS}} + \mathfrak{J}^{RN}_{\mathbb{U}_{BCFS}} + \mathfrak{J}^{IN}_{\mathbb{U}_{BCFS}}\right), \quad \mathcal{S}(\mathbb{U}_{BCFS}) \in [0,1] \quad (2)$$

$$\mathcal{H}(\mathbb{U}_{BCFS}) = \frac{\mathfrak{J}^{RP}_{\mathbb{U}_{BCFS}} + \mathfrak{J}^{IP}_{\mathbb{U}_{BCFS}} - \mathfrak{J}^{RN}_{\mathbb{U}_{BCFS}} - \mathfrak{J}^{IN}_{\mathbb{U}_{BCFS}}}{4}, \mathcal{H}(\mathbb{U}_{BCFS}) \in [0,1] \quad (3)$$

From Eq. (2) and (3), there are underneath outcomes

1. When $\mathcal{S}(\mathbb{U}_{BCFS-1}) < \mathcal{S}(\mathbb{U}_{BCFS-2})$, it means $\mathbb{U}_{BCFS-1} < \mathbb{U}_{BCFS-2}$;
2. When $\mathcal{S}(\mathbb{U}_{BCFS-1}) > \mathcal{S}(\mathbb{U}_{BCFS-2})$, it means $\mathbb{U}_{BCFS-1} > \mathbb{U}_{BCFS-2}$;
3. When $\mathcal{S}(\mathbb{U}_{BCFS-1}) = \mathcal{S}(\mathbb{U}_{BCFS-2})$, it means
   [1] When $\mathcal{H}(\mathbb{U}_{BCFS-1}) < \mathcal{H}(\mathbb{U}_{BCFS-2})$, it means $\mathbb{U}_{BCFS-1} < \mathbb{U}_{BCFS-2}$;
   [2] When $\mathcal{H}(\mathbb{U}_{BCFS-1}) > \mathcal{H}(\mathbb{U}_{BCFS-2})$, it means $\mathbb{U}_{BCFS-1} > \mathbb{U}_{BCFS-2}$;
   [3] When $\mathcal{H}(\mathbb{U}_{BCFS-1}) = \mathcal{H}(\mathbb{U}_{BCFS-2})$, it means $\mathbb{U}_{BCFS-1} = \mathbb{U}_{BCFS-2}$.

Definition 3: [41] Consider $\mathbb{U}_{BCFS-1} = \left(\mathfrak{J}^{P}_{\mathbb{U}_{BCFS-1}}, \mathfrak{J}^{N}_{\mathbb{U}_{BCFS-1}}\right) = \left(\mathfrak{J}^{RP}_{\mathbb{U}_{BCFS-1}} + \iota\, \mathfrak{J}^{IP}_{\mathbb{U}_{BCFS-1}}, \mathfrak{J}^{RN}_{\mathbb{U}_{BCFS-1}} + \iota\, \mathfrak{J}^{IN}_{\mathbb{U}_{BCFS-1}}\right)$ and $\mathbb{U}_{BCFS-2} = \left(\mathfrak{J}^{P}_{\mathbb{U}_{BCFS-2}}, \mathfrak{J}^{N}_{\mathbb{U}_{BCFS-2}}\right) = \left(\mathfrak{J}^{RP}_{\mathbb{U}_{BCFS-2}} + \iota\, \mathfrak{J}^{IP}_{\mathbb{U}_{BCFS-2}}, \mathfrak{J}^{RN}_{\mathbb{U}_{BCFS-2}} + \iota\, \mathfrak{J}^{IN}_{\mathbb{U}_{BCFS-2}}\right)$ are two BCFNs, with $\mathfrak{W} \geq 0$, then

$$\mathbb{U}_{BCFS-1} \oplus \mathbb{U}_{BCFS-2} = \begin{pmatrix} \mathfrak{J}^{RP}_{\mathbb{U}_{BCFS-1}} + \mathfrak{J}^{RP}_{\mathbb{U}_{BCFS-2}} - \mathfrak{J}^{RP}_{\mathbb{U}_{BCFS-1}}\mathfrak{J}^{RP}_{\mathbb{U}_{BCFS-2}} \\ +\iota\left(\mathfrak{J}^{IP}_{\mathbb{U}_{BCFS-1}} + \mathfrak{J}^{IP}_{\mathbb{U}_{BCFS-2}} - \mathfrak{J}^{IP}_{\mathbb{U}_{BCFS-1}}\mathfrak{J}^{IP}_{\mathbb{U}_{BCFS-2}}\right), \\ -\left(\mathfrak{J}^{RN}_{\mathbb{U}_{BCFS-1}}\mathfrak{J}^{RN}_{\mathbb{U}_{BCFS-2}}\right) + \iota\left(-\left(\mathfrak{J}^{IN}_{\mathbb{U}_{BCFS-1}}\mathfrak{J}^{IN}_{\mathbb{U}_{BCFS-2}}\right)\right) \end{pmatrix} \quad (4)$$

$$\mathbb{U}_{BCFS-1} \otimes \mathbb{U}_{BCFS-2} = \begin{pmatrix} \mathfrak{J}^{RP}_{\mathbb{U}_{BCFS-1}}\mathfrak{J}^{RP}_{\mathbb{U}_{BCFS-2}} + \iota\, \mathfrak{J}^{IP}_{\mathbb{U}_{BCFS-1}}\mathfrak{J}^{IP}_{\mathbb{U}_{BCFS-2}}, \\ \mathfrak{J}^{RN}_{\mathbb{U}_{BCFS-1}} + \mathfrak{J}^{RN}_{\mathbb{U}_{BCFS-2}} + \mathfrak{J}^{RN}_{\mathbb{U}_{BCFS-1}}\mathfrak{J}^{RN}_{\mathbb{U}_{BCFS-2}} \\ \iota\left(\mathfrak{J}^{IN}_{\mathbb{U}_{BCFS-1}} + \mathfrak{J}^{IN}_{\mathbb{U}_{BCFS-2}} + \mathfrak{J}^{IN}_{\mathbb{U}_{BCFS-1}}\mathfrak{J}^{IN}_{\mathbb{U}_{BCFS-2}}\right) \end{pmatrix} \quad (5)$$

$$\mathfrak{W}\mathbb{U}_{BCFS-1} = \begin{pmatrix} 1 - \left(1 - \mathfrak{J}^{RP}_{\mathbb{U}_{BCFS-1}}\right)^{\mathfrak{W}} + \iota\left(1 - \left(1 - \mathfrak{J}^{IP}_{\mathbb{U}_{BCFS-1}}\right)^{\mathfrak{W}}\right), \\ -\left|\mathfrak{J}^{RN}_{\mathbb{U}_{BCFS-1}}\right|^{\mathfrak{W}} + \iota\left(-\left|\mathfrak{J}^{IN}_{\mathbb{U}_{BCFS-1}}\right|^{\mathfrak{W}}\right) \end{pmatrix} \quad (6)$$

$$\mathbb{U}^{\mathfrak{W}}_{BCFS-1} = \begin{pmatrix} \left(\mathfrak{J}^{RP}_{\mathbb{U}_{BCFS-1}}\right)^{\mathfrak{W}} + \iota\left(\mathfrak{J}^{IP}_{\mathbb{U}_{BCFS-1}}\right)^{\mathfrak{W}}, \\ -1 + \left(1 + \mathfrak{J}^{RN}_{\mathbb{U}_{BCFS-1}}\right)^{\mathfrak{W}} + \iota\left(-1 + \left(1 + \mathfrak{J}^{IN}_{\mathbb{U}_{BCFS-1}}\right)^{\mathfrak{W}}\right) \end{pmatrix} \quad (7)$$

Definition 4: [47] Consider $\mathbb{U}_{\mathfrak{RL}-\varrho}, \varrho = 1, 2, \dots, \mathfrak{k}$ as a collection of classical numbers, then the PA operators originated as

$$PA(\mathbb{U}_{\mathfrak{RL}-1}, \mathbb{U}_{\mathfrak{RL}-2}, \dots, \mathbb{U}_{\mathfrak{RL}-\mathfrak{k}}) = \sum_{\varrho=1}^{\mathfrak{k}} \left(\frac{1 + \Upsilon(\mathbb{U}_{\mathfrak{RL}-\varrho})}{\sum_{\mathfrak{d}=1}^{\mathfrak{k}}\left(1 + \Upsilon(\mathbb{U}_{\mathfrak{RL}-\mathfrak{d}})\right)} \mathbb{U}_{\mathfrak{RL}-\varrho}\right) \quad (8)$$

$\Upsilon(\mathbb{U}_{\mathfrak{RL}-\mathfrak{d}}) = \sum_{\substack{\mathfrak{d}=1 \\ \mathfrak{d}\neq\varrho}}^{\mathfrak{k}} Supp(\mathbb{U}_{\mathfrak{RL}-\varrho}, \mathbb{U}_{\mathfrak{RL}-\mathfrak{d}})$, where $Supp(\mathbb{U}_{\mathfrak{RL}-\varrho}, \mathbb{U}_{\mathfrak{RL}-\mathfrak{d}})$ holds the following properties

1. $Supp(\mathbb{U}_{\mathfrak{RL}-\varrho}, \mathbb{U}_{\mathfrak{RL}-\mathfrak{d}}) \in [0, 1]$
2. $Supp(\mathbb{U}_{\mathfrak{RL}-\varrho}, \mathbb{U}_{\mathfrak{RL}-\mathfrak{d}}) = Supp(\mathbb{U}_{\mathfrak{RL}-\mathfrak{d}}, \mathbb{U}_{\mathfrak{RL}-\varrho})$
3. If $\mathfrak{D}_S(\mathbb{U}_{\mathfrak{RL}-\varrho}, \mathbb{U}_{\mathfrak{RL}-\mathfrak{d}}) < \mathfrak{D}_S\left(\mathbb{U}_{\mathfrak{RL}-\varrho'}, \mathbb{U}_{\mathfrak{RL}-\mathfrak{d}'}\right)$, then $Supp(\mathbb{U}_{\mathfrak{RL}-\varrho}, \mathbb{U}_{\mathfrak{RL}-\mathfrak{d}}) > Supp\left(\mathbb{U}_{\mathfrak{RL}-\varrho'}, \mathbb{U}_{\mathfrak{RL}-\mathfrak{d}'}\right)$, where $\mathfrak{D}_S(\mathbb{U}_{\mathfrak{RL}-\varrho}, \mathbb{U}_{\mathfrak{RL}-\mathfrak{d}})$ signifies the distance between $\mathbb{U}_{\mathfrak{RL}-\varrho}$ and $\mathbb{U}_{\mathfrak{RL}-\mathfrak{d}}$.

and $Supp(\mathbb{U}_{\mathfrak{RL}-\varrho}, \mathbb{U}_{\mathfrak{RL}-\mathfrak{d}})$ is known as the support degree for $\mathbb{U}_{\mathfrak{RL}-\varrho}$ and $\mathbb{U}_{\mathfrak{RL}-\mathfrak{d}}$.

Definition 5: [48] Consider $\mathbb{U}_{\Re\mathfrak{L}-\cancel{c}}, \cancel{c} = 1, 2, \dots, \text{ł}$ as a collection of classical numbers and $\mathfrak{Z}_{VP} = (\mathfrak{Z}_{VP-1}, \mathfrak{Z}_{VP-2}, \dots \mathfrak{Z}_{VP-\text{ł}}) \in \Re^{\text{ł}}$ as vector parameters, then the MM operators originated as

$$MM^{\mathfrak{Z}_{VP}}(\mathbb{U}_{\Re\mathfrak{L}-1}, \mathbb{U}_{\Re\mathfrak{L}-2}, \dots, \mathbb{U}_{\Re\mathfrak{L}-\text{ł}}) = \left(\frac{1}{\text{ł}!}\sum_{\mathcal{G}\in \mathrm{P}_{\text{ł}}}\prod_{\cancel{c}=1}^{\text{ł}}\left(\mathbb{U}_{\Re\mathfrak{L}-\mathcal{G}(\cancel{c})}\right)^{\mathfrak{Z}_{VP-\cancel{c}}}\right)^{\frac{1}{\sum_{\cancel{c}=1}^{\text{ł}}\mathfrak{Z}_{VP-\cancel{c}}}} \quad (9)$$

where $\mathrm{P}_{\text{ł}}$ would signify the collection of permutations, $\mathcal{G}(\cancel{c})$ would be any permutation of $(1, 2, \dots, \text{ł})$.

3. | POWER MUIRHEAD MEAN OPERATORS BASED ON BCFS

In this Section, we define BCFPMM, BCFWPMM, BCFPDMM, and BCFWPDMM operators and the properties of these operators.

Definition 6: Consider $\mathbb{U}_{BCFS-\cancel{c}} = \left(\mathfrak{I}^{P}_{\mathbb{U}_{BCFS-\cancel{c}}}, \mathfrak{I}^{N}_{\mathbb{U}_{BCFS-\cancel{c}}}\right) = \left(\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\cancel{c}}} + \iota\, \mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\cancel{c}}}, \mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\cancel{c}}} + \iota\, \mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\cancel{c}}}\right), \cancel{c} = 1, 2, \dots, \text{ł}$ as a collection of BCFNs and $\mathfrak{Z}_{VP} = (\mathfrak{Z}_{VP-1}, \mathfrak{Z}_{VP-2}, \dots \mathfrak{Z}_{VP-\text{ł}}) \in \Re^{\text{ł}}$ as a vector of parameters, then

$$BCFPMM^{\mathfrak{Z}_{VP}}(\mathbb{U}_{BCFS-1}, \mathbb{U}_{BCFS-2}, \dots, \mathbb{U}_{BCFS-\text{ł}}) = \left(\frac{1}{\text{ł}!}\sum_{\mathcal{G}\in \mathrm{P}_{\text{ł}}}\prod_{\cancel{c}=1}^{\text{ł}}\left(\frac{\text{ł}\left(1 + \Upsilon\left(\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}\right)\right)}{\sum_{\text{đ}=1}^{\text{ł}}\left(1 + \Upsilon(\mathbb{U}_{BCFS-\text{đ}})\right)}\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}\right)^{\mathfrak{Z}_{VP-\cancel{c}}}\right)^{\frac{1}{\sum_{\cancel{c}=1}^{\text{ł}}\mathfrak{Z}_{VP-\cancel{c}}}} \quad (10)$$

Would say to be a BCFPMM operator, where $\mathrm{P}_{\text{ł}}$ would signify the collection of permutations, $\mathcal{G}(\cancel{c})$ would be any permutation of $(1, 2, \dots, \text{ł})$ and $\Upsilon(\mathbb{U}_{BCFS-\text{đ}}) = \sum_{\substack{\text{đ}=1 \\ \text{đ}\neq\cancel{c}}}^{\text{ł}} Supp(\mathbb{U}_{BCFS-\cancel{c}}, \mathbb{U}_{BCFS-\text{đ}})$, where $Supp(\mathbb{U}_{BCFS-\cancel{c}}, \mathbb{U}_{BCFS-\text{đ}})$ holds the properties underneath

1. $Supp(\mathbb{U}_{BCFS-\cancel{c}}, \mathbb{U}_{BCFS-\text{đ}}) \in [0, 1]$
2. $Supp(\mathbb{U}_{BCFS-\cancel{c}}, \mathbb{U}_{BCFS-\text{đ}}) = Supp(\mathbb{U}_{BCFS-z}, \mathbb{U}_{BCFS-\cancel{c}})$
3. If $\mathfrak{D}_s(\mathbb{U}_{BCFS-\cancel{c}}, \mathbb{U}_{BCFS-\text{đ}}) < \mathfrak{D}_s\left(\mathbb{U}_{BCFS-\cancel{c}'}, \mathbb{U}_{BCFS-\text{đ}'}\right)$, then $Supp(\mathbb{U}_{BCFS-\cancel{c}}, \mathbb{U}_{BCFS-\text{đ}}) > Supp\left(\mathbb{U}_{BCFS-\cancel{c}'}, \mathbb{U}_{BCFS-\text{đ}'}\right)$, where $\mathfrak{D}_s(\mathbb{U}_{BCFS-\cancel{c}}, \mathbb{U}_{BCFS-\text{đ}})$ signifies the distance between $\mathbb{U}_{BCFS-\cancel{c}}$ and $\mathbb{U}_{BCFS-\text{đ}}$.

and $Supp(\mathbb{U}_{BCFS-\cancel{c}}, \mathbb{U}_{BCFS-\text{đ}})$ is known as the support degree for $\mathbb{U}_{BCFS-\cancel{c}}$ and $\mathbb{U}_{BCFS-\text{đ}}$.

To simply Eq. (10) we consider $\frac{\left(1+\Upsilon\left(\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}\right)\right)}{\sum_{\text{đ}=1}^{\text{ł}}\left(1+\Upsilon(\mathbb{U}_{BCFS-\text{đ}})\right)} = \mathfrak{F}_{\cancel{c}}$, then Eq. (10) converted to the following form

$$BCFPMM^{\mathfrak{Z}_{VP}}(\mathbb{U}_{BCFS-1}, \mathbb{U}_{BCFS-2}, \dots, \mathbb{U}_{BCFS-\text{ł}}) = \left(\frac{1}{\text{ł}!}\sum_{\mathcal{G}\in \mathrm{P}_{\text{ł}}}\prod_{\cancel{c}=1}^{\text{ł}}\left(\text{ł}\mathfrak{F}_{\cancel{c}}\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}\right)^{\mathfrak{Z}_{VP-\cancel{c}}}\right)^{\frac{1}{\sum_{\cancel{c}=1}^{\text{ł}}\mathfrak{Z}_{VP-\cancel{c}}}} \quad (11)$$

and for appropriateness, we can consider that $(\mathfrak{F}_1, \mathfrak{F}_2, \dots, \mathfrak{F}_{\text{ł}})^T$ would be the power weight vector with the condition that $\mathfrak{F}_{\cancel{c}} \in [0, 1]$ and $\sum_{\cancel{c}=1}^{\text{ł}}\mathfrak{F}_{\cancel{c}} = 1$.

Theorem 1: Consider $\mathbb{U}_{BCFS-\not{c}} = \left(\mathfrak{J}^{P}_{\mathbb{U}_{BCFS-\not{c}}}, \mathfrak{J}^{N}_{\mathbb{U}_{BCFS-\not{c}}}\right) = \left(\mathfrak{J}^{RP}_{\mathbb{U}_{BCFS-\not{c}}} + \iota\, \mathfrak{J}^{IP}_{\mathbb{U}_{BCFS-\not{c}}}, \mathfrak{J}^{RN}_{\mathbb{U}_{BCFS-\not{c}}} + \iota\, \mathfrak{J}^{IN}_{\mathbb{U}_{BCFS-\not{c}}}\right), \not{c} = 1, 2, \dots, \text{ł}$ as a collection of BCFNs, then the result after aggregating the BCFNs with the assistance of Eq. (11) is yet again a BCFN

$$BCFPMM^{3_{VP}}\left(\mathbb{U}_{BCFS-1}, \mathbb{U}_{BCFS-2}, \dots, \mathbb{U}_{BCFS-\text{ł}}\right)$$

$$= \begin{pmatrix} \left(1 - \left(\prod_{\mathcal{G}\in \mathrm{P}_{\text{ł}}}\left(1 - \prod_{\not{c}=1}^{\text{ł}}\left(1 - \left(1 - \mathfrak{J}^{RP}_{\mathbb{U}_{BCFS-\mathcal{G}(\not{c})}}\right)^{\text{ł}\digamma_{\not{c}}}\right)^{3_{VP-\not{c}}}\right)\right)^{\frac{1}{\text{ł}!}}\right)^{\frac{1}{\sum_{\not{c}=1}^{\text{ł}} 3_{VP-\not{c}}}} \\ \left(1 - \left(\prod_{\mathcal{G}\in \mathrm{P}_{\text{ł}}}\left(1 - \prod_{\not{c}=1}^{\text{ł}}\left(1 - \left(1 - \mathfrak{J}^{IP}_{\mathbb{U}_{BCFS-\mathcal{G}(\not{c})}}\right)^{\text{ł}\digamma_{\not{c}}}\right)^{3_{VP-\not{c}}}\right)\right)^{\frac{1}{\text{ł}!}}\right)^{\frac{1}{\sum_{\not{c}=1}^{\text{ł}} 3_{VP-\not{c}}}}, \\ -1 + \left(1 - \left|-\prod_{\mathcal{G}\in \mathrm{P}_{\text{ł}}}\left(-1 + \prod_{\not{c}=1}^{\text{ł}}\left(1 - \left|\mathfrak{J}^{RN}_{\mathbb{U}_{BCFS-\mathcal{G}(\not{c})}}\right|^{\text{ł}\digamma_{\not{c}}}\right)^{3_{VP-\not{c}}}\right)\right|^{\frac{1}{\text{ł}!}}\right)^{\frac{1}{\sum_{\not{c}=1}^{\text{ł}} 3_{VP-\not{c}}}} \\ -1 + \left(1 - \left|-\prod_{\mathcal{G}\in \mathrm{P}_{\text{ł}}}\left(-1 + \prod_{\not{c}=1}^{\text{ł}}\left(1 - \left|\mathfrak{J}^{IN}_{\mathbb{U}_{BCFS-\mathcal{G}(\not{c})}}\right|^{\text{ł}\digamma_{\not{c}}}\right)^{3_{VP-\not{c}}}\right)\right|^{\frac{1}{\text{ł}!}}\right)^{\frac{1}{\sum_{\not{c}=1}^{\text{ł}} 3_{VP-\not{c}}}} \end{pmatrix} \quad (12)$$

Proof: With the help of Def (3), we have

$$\text{ł}\digamma_{\not{c}}\mathbb{U}_{BCFS-\mathcal{G}(\not{c})} = \begin{pmatrix} 1 - \left(1 - \mathfrak{J}^{RP}_{\mathbb{U}_{BCFS-\mathcal{G}(\not{c})}}\right)^{\text{ł}\digamma_{\not{c}}} + \iota\left(1 - \left(1 - \mathfrak{J}^{IP}_{\mathbb{U}_{BCFS-\mathcal{G}(\not{c})}}\right)^{\text{ł}\digamma_{\not{c}}}\right), \\ -\left|\mathfrak{J}^{RN}_{\mathbb{U}_{BCFS-\mathcal{G}(\not{c})}}\right|^{\text{ł}\digamma_{\not{c}}} + \iota\left(-\left|\mathfrak{J}^{IN}_{\mathbb{U}_{BCFS-\mathcal{G}(\not{c})}}\right|^{\text{ł}\digamma_{\not{c}}}\right) \end{pmatrix}$$

$$\left(\text{ł}\digamma_{\not{c}}\mathbb{U}_{BCFS-\mathcal{G}(\not{c})}\right)^{3_{VP-\not{c}}} = \begin{pmatrix} \left(1 - \left(1 - \mathfrak{J}^{RP}_{\mathbb{U}_{BCFS-\mathcal{G}(\not{c})}}\right)^{\text{ł}\digamma_{\not{c}}}\right)^{3_{VP-\not{c}}} + \iota\left(1 - \left(1 - \mathfrak{J}^{IP}_{\mathbb{U}_{BCFS-\mathcal{G}(\not{c})}}\right)^{\text{ł}\digamma_{\not{c}}}\right)^{3_{VP-\not{c}}}, \\ -1 + \left(1 - \left|\mathfrak{J}^{RN}_{\mathbb{U}_{BCFS-\mathcal{G}(\not{c})}}\right|^{\text{ł}\digamma_{\not{c}}}\right)^{3_{VP-\not{c}}} + \iota\left(-1 + \left(1 - \left|\mathfrak{J}^{IN}_{\mathbb{U}_{BCFS-\mathcal{G}(\not{c})}}\right|^{\text{ł}\digamma_{\not{c}}}\right)^{3_{VP-\not{c}}}\right) \end{pmatrix}$$

then,

$$\prod_{\not{c}=1}^{\text{ł}}\left(\text{ł}\digamma_{\not{c}}\mathbb{U}_{BCFS-\mathcal{G}(\not{c})}\right)^{3_{VP-\not{c}}}$$

$$= \begin{pmatrix} \prod_{\not{c}=1}^{\text{ł}}\left(1 - \left(1 - \mathfrak{J}^{RP}_{\mathbb{U}_{BCFS-\mathcal{G}(\not{c})}}\right)^{\text{ł}\digamma_{\not{c}}}\right)^{3_{VP-\not{c}}} + \iota\prod_{\not{c}=1}^{\text{ł}}\left(1 - \left(1 - \mathfrak{J}^{IP}_{\mathbb{U}_{BCFS-\mathcal{G}(\not{c})}}\right)^{\text{ł}\digamma_{\not{c}}}\right)^{3_{VP-\not{c}}}, \\ -1 + \prod_{\not{c}=1}^{\text{ł}}\left(1 - \left|\mathfrak{J}^{RN}_{\mathbb{U}_{BCFS-\mathcal{G}(\not{c})}}\right|^{\text{ł}\digamma_{\not{c}}}\right)^{3_{VP-\not{c}}} + \iota\left(-1 + \prod_{\not{c}=1}^{\text{ł}}\left(1 - \left|\mathfrak{J}^{IN}_{\mathbb{U}_{BCFS-\mathcal{G}(\not{c})}}\right|^{\text{ł}\digamma_{\not{c}}}\right)^{3_{VP-\not{c}}}\right) \end{pmatrix}$$

and

$$\sum_{\mathcal{G}\in\mathcal{P}_{\text{ł}}}\prod_{\text{ȼ}=1}^{\text{ł}}\left(\text{łƑ}_{\text{ȼ}}\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}\right)^{3_{VP-\text{ȼ}}}=\left(\begin{array}{c}1-\prod_{\mathcal{G}\in\mathcal{P}_{\text{ł}}}\left(1-\prod_{\text{ȼ}=1}^{\text{ł}}\left(1-\left(1-\mathfrak{I}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}^{RP}\right)^{\text{łƑ}_{\text{ȼ}}}\right)^{3_{VP-\text{ȼ}}}\right)\\ +\iota\left(1-\prod_{\mathcal{G}\in\mathcal{P}_{\text{ł}}}\left(1-\prod_{\text{ȼ}=1}^{\text{ł}}\left(1-\left(1-\mathfrak{I}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}^{IP}\right)^{\text{łƑ}_{\text{ȼ}}}\right)^{3_{VP-\text{ȼ}}}\right)\right),\\ -\prod_{\mathcal{G}\in\mathcal{P}_{\text{ł}}}\left(-1+\prod_{\text{ȼ}=1}^{\text{ł}}\left(1-\left|\mathfrak{I}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}^{RN}\right|^{\text{łƑ}_{\text{ȼ}}}\right)^{3_{VP-\text{ȼ}}}\right)\\ -\prod_{\mathcal{G}\in\mathcal{P}_{\text{ł}}}\left(-1+\prod_{\text{ȼ}=1}^{\text{ł}}\left(1-\left|\mathfrak{I}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}^{IN}\right|^{\text{łƑ}_{\text{ȼ}}}\right)^{3_{VP-\text{ȼ}}}\right)\end{array}\right)$$

next,

$$\frac{1}{\text{ł}!}\sum_{\mathcal{G}\in\mathcal{P}_{\text{ł}}}\prod_{\text{ȼ}=1}^{\text{ł}}\left(\text{łƑ}_{\text{ȼ}}\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}\right)^{3_{VP-\text{ȼ}}}=\left(\begin{array}{c}1-\left(\prod_{\mathcal{G}\in\mathcal{P}_{\text{ł}}}\left(1-\prod_{\text{ȼ}=1}^{\text{ł}}\left(1-\left(1-\mathfrak{I}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}^{RP}\right)^{\text{łƑ}_{\text{ȼ}}}\right)^{3_{VP-\text{ȼ}}}\right)\right)^{\frac{1}{\text{ł}!}}\\ 1-\left(\prod_{\mathcal{G}\in\mathcal{P}_{\text{ł}}}\left(1-\prod_{\text{ȼ}=1}^{\text{ł}}\left(1-\left(1-\mathfrak{I}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}^{IP}\right)^{\text{łƑ}_{\text{ȼ}}}\right)^{3_{VP-\text{ȼ}}}\right)\right)^{\frac{1}{\text{ł}!}},\\ -\left|-\prod_{\mathcal{G}\in\mathcal{P}_{\text{ł}}}\left(-1+\prod_{\text{ȼ}=1}^{\text{ł}}\left(1-\left|\mathfrak{I}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}^{RN}\right|^{\text{łƑ}_{\text{ȼ}}}\right)^{3_{VP-\text{ȼ}}}\right)\right|^{\frac{1}{\text{ł}!}}\\ -\left|-\prod_{\mathcal{G}\in\mathcal{P}_{\text{ł}}}\left(-1+\prod_{\text{ȼ}=1}^{\text{ł}}\left(1-\left|\mathfrak{I}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}^{IN}\right|^{\text{łƑ}_{\text{ȼ}}}\right)^{3_{VP-\text{ȼ}}}\right)\right|^{\frac{1}{\text{ł}!}}\end{array}\right)$$

thus,

$$\left(\frac{1}{\text{ƚ}!}\sum_{\mathcal{G}\in \text{Ᵽ}_{\text{ƚ}}}\prod_{\text{ȼ}=1}^{\text{ƚ}}\left(\text{ƚƑ}_{\text{ȼ}}\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}\right)^{3_{VP-\text{ȼ}}}\right)^{\frac{1}{\sum_{\text{ȼ}=1}^{\text{ƚ}}3_{VP-\text{ȼ}}}}$$

$$= \begin{pmatrix} \left(1-\left(\prod_{\mathcal{G}\in \text{Ᵽ}_{\text{ƚ}}}\left(1-\prod_{\text{ȼ}=1}^{\text{ƚ}}\left(1-\left(1-\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}\right)^{\text{ƚƑ}_{\text{ȼ}}}\right)^{3_{VP-\text{ȼ}}}\right)\right)^{\frac{1}{\text{ƚ}!}}\right)^{\frac{1}{\sum_{\text{ȼ}=1}^{\text{ƚ}}3_{VP-\text{ȼ}}}} \\ \left(1-\left(\prod_{\mathcal{G}\in \text{Ᵽ}_{\text{ƚ}}}\left(1-\prod_{\text{ȼ}=1}^{\text{ƚ}}\left(1-\left(1-\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}\right)^{\text{ƚƑ}_{\text{ȼ}}}\right)^{3_{VP-\text{ȼ}}}\right)\right)^{\frac{1}{\text{ƚ}!}}\right)^{\frac{1}{\sum_{\text{ȼ}=1}^{\text{ƚ}}3_{VP-\text{ȼ}}}}, \\ -1+\left(1-\left|-\prod_{\mathcal{G}\in \text{Ᵽ}_{\text{ƚ}}}\left(-1+\prod_{\text{ȼ}=1}^{\text{ƚ}}\left(1-\left|\mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}\right|^{\text{ƚƑ}_{\text{ȼ}}}\right)^{3_{VP-\text{ȼ}}}\right)\right|^{\frac{1}{\text{ƚ}!}}\right)^{\frac{1}{\sum_{\text{ȼ}=1}^{\text{ƚ}}3_{VP-\text{ȼ}}}} \\ -1+\left(1-\left|-\prod_{\mathcal{G}\in \text{Ᵽ}_{\text{ƚ}}}\left(-1+\prod_{\text{ȼ}=1}^{\text{ƚ}}\left(1-\left|\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}\right|^{\text{ƚƑ}_{\text{ȼ}}}\right)^{3_{VP-\text{ȼ}}}\right)\right|^{\frac{1}{\text{ƚ}!}}\right)^{\frac{1}{\sum_{\text{ȼ}=1}^{\text{ƚ}}3_{VP-\text{ȼ}}}} \end{pmatrix}$$

consequently,

$$BCFPMM^{3_{VP}}\left(\mathbb{U}_{BCFS-1},\mathbb{U}_{BCFS-2},\dots,\mathbb{U}_{BCFS-\text{ƚ}}\right)$$

$$= \begin{pmatrix} \left(1-\left(\prod_{\mathcal{G}\in \text{Ᵽ}_{\text{ƚ}}}\left(1-\prod_{\text{ȼ}=1}^{\text{ƚ}}\left(1-\left(1-\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}\right)^{\text{ƚƑ}_{\text{ȼ}}}\right)^{3_{VP-\text{ȼ}}}\right)\right)^{\frac{1}{\text{ƚ}!}}\right)^{\frac{1}{\sum_{\text{ȼ}=1}^{\text{ƚ}}3_{VP-\text{ȼ}}}} \\ \left(1-\left(\prod_{\mathcal{G}\in \text{Ᵽ}_{\text{ƚ}}}\left(1-\prod_{\text{ȼ}=1}^{\text{ƚ}}\left(1-\left(1-\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}\right)^{\text{ƚƑ}_{\text{ȼ}}}\right)^{3_{VP-\text{ȼ}}}\right)\right)^{\frac{1}{\text{ƚ}!}}\right)^{\frac{1}{\sum_{\text{ȼ}=1}^{\text{ƚ}}3_{VP-\text{ȼ}}}}, \\ -1+\left(1-\left|-\prod_{\mathcal{G}\in \text{Ᵽ}_{\text{ƚ}}}\left(-1+\prod_{\text{ȼ}=1}^{\text{ƚ}}\left(1-\left|\mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}\right|^{\text{ƚƑ}_{\text{ȼ}}}\right)^{3_{VP-\text{ȼ}}}\right)\right|^{\frac{1}{\text{ƚ}!}}\right)^{\frac{1}{\sum_{\text{ȼ}=1}^{\text{ƚ}}3_{VP-\text{ȼ}}}} \\ -1+\left(1-\left|-\prod_{\mathcal{G}\in \text{Ᵽ}_{\text{ƚ}}}\left(-1+\prod_{\text{ȼ}=1}^{\text{ƚ}}\left(1-\left|\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}\right|^{\text{ƚƑ}_{\text{ȼ}}}\right)^{3_{VP-\text{ȼ}}}\right)\right|^{\frac{1}{\text{ƚ}!}}\right)^{\frac{1}{\sum_{\text{ȼ}=1}^{\text{ƚ}}3_{VP-\text{ȼ}}}} \end{pmatrix}$$

Theorem 2: (Idempotency) Consider $\mathbb{U}_{BCFS-\not{c}} = \left(\Im^{P}_{\mathbb{U}_{BCFS-\not{c}}}, \Im^{N}_{\mathbb{U}_{BCFS-\not{c}}}\right) = \left(\Im^{RP}_{\mathbb{U}_{BCFS-\not{c}}} + \iota\,\Im^{IP}_{\mathbb{U}_{BCFS-\not{c}}}, \Im^{RN}_{\mathbb{U}_{BCFS-\not{c}}} + \iota\,\Im^{IN}_{\mathbb{U}_{BCFS-\not{c}}}\right), \not{c} = 1, 2, \dots, ł$ as a collection of BCFNs and $\mathbb{U}_{BCFS-\not{c}} = \mathbb{U}_{BCFS}\ \forall\ \not{c}$, then

$$BCFPMM^{\mathfrak{Z}_{VP}}(\mathbb{U}_{BCFS-1}, \mathbb{U}_{BCFS-2}, \dots, \mathbb{U}_{BCFS-ł}) = \mathbb{U}_{BCFS}$$

Theorem 3: (Monotonicity) Consider $\mathbb{U}_{BCFS-\not{c}} = \left(\Im^{P}_{\mathbb{U}_{BCFS-\not{c}}}, \Im^{N}_{\mathbb{U}_{BCFS-\not{c}}}\right) = \left(\Im^{RP}_{\mathbb{U}_{BCFS-\not{c}}} + \iota\,\Im^{IP}_{\mathbb{U}_{BCFS-\not{c}}}, \Im^{RN}_{\mathbb{U}_{BCFS-\not{c}}} + \iota\,\Im^{IN}_{\mathbb{U}_{BCFS-\not{c}}}\right)$, and $\mathbb{U}^{\#}_{BCFS-\not{c}} = \left(\Im^{P}_{\mathbb{U}^{\#}_{BCFS-\not{c}}}, \Im^{N}_{\mathbb{U}^{\#}_{BCFS-\not{c}}}\right) = \left(\Im^{RP}_{\mathbb{U}^{\#}_{BCFS-\not{c}}} + \iota\,\Im^{IP}_{\mathbb{U}^{\#}_{BCFS-\not{c}}}, \Im^{RN}_{\mathbb{U}^{\#}_{BCFS-\not{c}}} + \iota\,\Im^{IN}_{\mathbb{U}^{\#}_{BCFS-\not{c}}}\right), \not{c} = 1, 2, \dots, ł$ as two collections of BCFNs and $\mathbb{U}_{BCFS-\not{c}} \le \mathbb{U}^{\#}_{BCFS-\not{c}}$ that is, $\Im^{RP}_{\mathbb{U}_{BCFS-\not{c}}} \le \Im^{RP}_{\mathbb{U}^{\#}_{BCFS-\not{c}}}$, $\Im^{IP}_{\mathbb{U}_{BCFS-\not{c}}} \le \Im^{IP}_{\mathbb{U}^{\#}_{BCFS-\not{c}}}$, $\Im^{RN}_{\mathbb{U}_{BCFS-\not{c}}} \le \Im^{RN}_{\mathbb{U}^{\#}_{BCFS-\not{c}}}$ and $\Im^{IN}_{\mathbb{U}_{BCFS-\not{c}}} \le \Im^{IN}_{\mathbb{U}^{\#}_{BCFS-\not{c}}}$, then

$$BCFPMM^{\mathfrak{Z}_{VP}}(\mathbb{U}_{BCFS-1}, \mathbb{U}_{BCFS-2}, \dots, \mathbb{U}_{BCFS-ł}) \le BCFPMM^{\mathfrak{Z}_{VP}}\left(\mathbb{U}^{\#}_{BCFS-1}, \mathbb{U}^{\#}_{BCFS-2}, \dots, \mathbb{U}^{\#}_{BCFS-ł}\right)$$

Theorem 4: (Boundedness) Consider $\mathbb{U}_{BCFS-\not{c}} = \left(\Im^{P}_{\mathbb{U}_{BCFS-\not{c}}}, \Im^{N}_{\mathbb{U}_{BCFS-\not{c}}}\right) = \left(\Im^{RP}_{\mathbb{U}_{BCFS-\not{c}}} + \iota\,\Im^{IP}_{\mathbb{U}_{BCFS-\not{c}}}, \Im^{RN}_{\mathbb{U}_{BCFS-\not{c}}} + \iota\,\Im^{IN}_{\mathbb{U}_{BCFS-\not{c}}}\right), \not{c} = 1, 2, \dots, ł$ as a collection of BCFNs and $\mathbb{U}^{-}_{BCFS} = \left(\min_{\not{c}}\left\{\Im^{RP}_{\mathbb{U}_{BCFS-\not{c}}}\right\} + \iota \min_{\not{c}}\left\{\Im^{IP}_{\mathbb{U}_{BCFS-\not{c}}}\right\}, \max_{\not{c}}\left\{\Im^{RN}_{\mathbb{U}_{BCFS-\not{c}}}\right\} + \iota \max_{\not{c}}\left\{\Im^{IN}_{\mathbb{U}_{BCFS-\not{c}}}\right\}\right)$, and $\mathbb{U}^{+}_{BCFS} = \left(\max_{\not{c}}\left\{\Im^{RP}_{\mathbb{U}_{BCFS-\not{c}}}\right\} + \iota \max_{\not{c}}\left\{\Im^{IP}_{\mathbb{U}_{BCFS-\not{c}}}\right\}, \min_{\not{c}}\left\{\Im^{RN}_{\mathbb{U}_{BCFS-\not{c}}}\right\} + \iota \min_{\not{c}}\left\{\Im^{IN}_{\mathbb{U}_{BCFS-\not{c}}}\right\}\right)$, then

$$\mathbb{U}^{-}_{BCFS} \le BCFPMM^{\mathfrak{Z}_{VP}}(\mathbb{U}_{BCFS-1}, \mathbb{U}_{BCFS-2}, \dots, \mathbb{U}_{BCFS-ł}) \le \mathbb{U}^{+}_{BCFS}$$

Underneath are the particular cases of the investigated BCFPMM operator

Case 1: For $\mathfrak{Z}_{VP} = (1, 0, 0, \dots, 0)$, we achieved the BCF power averaging (BCFPA) operator from the diagnosed BCFPMM operator, i.e.

$$BCFPMM^{(1,0,0,\dots,0)}(\mathbb{U}_{BCFS-1}, \mathbb{U}_{BCFS-2}, \dots, \mathbb{U}_{BCFS-ł}) = \left(\sum_{\not{c}=1}^{ł} \frac{\left(1 + \Upsilon(\mathbb{U}_{BCFS-\not{c}})\right)}{\sum_{d=1}^{ł}\left(1 + \Upsilon(\mathbb{U}_{BCFS-d})\right)} \mathbb{U}_{BCFS-\not{c}}\right)$$
$$= BCFPA(\mathbb{U}_{BCFS-1}, \mathbb{U}_{BCFS-2}, \dots, \mathbb{U}_{BCFS-ł})$$

Case 2: For $\mathfrak{Z}_{VP} = \left(\frac{1}{ł}, \frac{1}{ł}, \frac{1}{ł}, \dots, \frac{1}{ł}\right)$, we achieved the BCF power geometric (BCFPG) operator from the diagnosed BCFPMM operator, i.e.

$$BCFPMM^{\left(\frac{1}{ł},\frac{1}{ł},\frac{1}{ł},\dots,\frac{1}{ł}\right)}(\mathbb{U}_{BCFS-1}, \mathbb{U}_{BCFS-2}, \dots, \mathbb{U}_{BCFS-ł}) = \prod_{\not{c}=1}^{ł} (\mathbb{U}_{BCFS-\not{c}})^{\sum_{\not{c}=1}^{ł} \frac{\left(1+\Upsilon(\mathbb{U}_{BCFS-\not{c}})\right)}{\sum_{d=1}^{ł}\left(1+\Upsilon(\mathbb{U}_{BCFS-d})\right)}}$$
$$= BCFPG(\mathbb{U}_{BCFS-1}, \mathbb{U}_{BCFS-2}, \dots, \mathbb{U}_{BCFS-ł})$$

Case 3: For $\mathfrak{Z}_{VP} = (1, 1, 0, \dots, 0)$, we achieved the BCF power Bonferroni mean (BCPBM) operator (with $p = q = 1$) from the diagnosed BCFPMM operator.

Case 4: For $\mathfrak{Z}_{VP} = (1, 1, \dots 1, 0, 0, \dots, 0)$, we achieved the BCF power Maclaurin symmetric mean (BCPMSM) operator from the diagnosed BCFPMM operator.

Definition 7: Consider $\mathbb{U}_{BCFS-\not{c}} = \left(\Im^{P}_{\mathbb{U}_{BCFS-\not{c}}}, \Im^{N}_{\mathbb{U}_{BCFS-\not{c}}}\right) = \left(\Im^{RP}_{\mathbb{U}_{BCFS-\not{c}}} + \iota\,\Im^{IP}_{\mathbb{U}_{BCFS-\not{c}}}, \Im^{RN}_{\mathbb{U}_{BCFS-\not{c}}} + \iota\,\Im^{IN}_{\mathbb{U}_{BCFS-\not{c}}}\right), \not{c} = 1, 2, \dots, ł$ as a collection of BCFNs and $\mathfrak{Z}_{VP} = (\mathfrak{Z}_{VP-1}, \mathfrak{Z}_{VP-2}, \dots \mathfrak{Z}_{VP-ł}) \in \Re^{ł}$ as a vector of parameters, then

$$BCFWPMM^{3_{VP}}(\mathbb{U}_{BCFS-1},\mathbb{U}_{BCFS-2},\dots,\mathbb{U}_{BCFS-\text{ł}})$$
$$=\left(\frac{1}{\text{ł}!}\sum_{\mathcal{G}\in\text{Ƥ}_{\text{ł}}}\prod_{\text{ȼ}=1}^{\text{ł}}\left(\text{ł}\frac{w_{ww-\mathcal{G}(\text{ȼ})}\left(1+\text{Ƭ}\left(\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}\right)\right)}{\sum_{\text{đ}=1}^{\text{ł}}w_{ww-\text{đ}}\left(1+\text{Ƭ}\left(\mathbb{U}_{BCFS-\text{đ}}\right)\right)}\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}\right)^{3_{VP-\text{ȼ}}}\right)^{\frac{1}{\sum_{\text{ȼ}=1}^{\text{ł}}3_{VP-\text{ȼ}}}} \quad (13)$$

Would say to be a BCFWPMM operator, where $\text{Ƥ}_{\text{ł}}$ would signify the collection of permutations, $\mathcal{G}(\text{ȼ})$ would be any permutation of $(1,2,\dots,\text{ł})$, $\text{Ƭ}(\mathbb{U}_{BCFS-\text{đ}})=\sum_{\substack{\text{đ}=1\\ \text{đ}\neq\text{ȼ}}}^{\text{ł}}Supp(\mathbb{U}_{BCFS-\text{ȼ}},\mathbb{U}_{BCFS-\text{đ}})$, $Supp(\mathbb{U}_{BCFS-\text{ȼ}},\mathbb{U}_{BCFS-\text{đ}})$ is known as the support degree for $\mathbb{U}_{BCFS-\text{ȼ}}$ and $\mathbb{U}_{BCFS-\text{đ}}$ and $w_{ww}=(w_{ww-1},w_{ww-2},\dots,w_{ww-\text{ł}})$ would be the weight vector meet with condition that $\forall\,\text{ł}\;0\le w_{ww-\text{ȼ}}\le 1$ and $\sum_{\text{ȼ}=1}^{\text{ł}}w_{ww-\text{ȼ}}=1$.

Theorem 5: Consider $\mathbb{U}_{BCFS-\text{ȼ}}=\left(\Im^{P}_{\mathbb{U}_{BCFS-\text{ȼ}}},\Im^{N}_{\mathbb{U}_{BCFS-\text{ȼ}}}\right)=\left(\Im^{RP}_{\mathbb{U}_{BCFS-\text{ȼ}}}+\iota\,\Im^{IP}_{\mathbb{U}_{BCFS-\text{ȼ}}},\Im^{RN}_{\mathbb{U}_{BCFS-\text{ȼ}}}+\iota\,\Im^{IN}_{\mathbb{U}_{BCFS-\text{ȼ}}}\right),\text{ȼ}=1,2,\dots,\text{ł}$ as a collection of BCFNs, then the result after aggregating the BCFNs with the assistance of Eq. (13) is yet again a BCFN

$$BCFWPMM^{3_{VP}}(\mathbb{U}_{BCFS-1},\mathbb{U}_{BCFS-2},\dots,\mathbb{U}_{BCFS-\text{ł}})$$
$$=\left(\begin{array}{l}
\left(1-\left(\prod_{\mathcal{G}\in\text{Ƥ}_{\text{ł}}}\left(1-\prod_{\text{ȼ}=1}^{\text{ł}}\left(1-\left(1-\Im^{RP}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}\right)^{\text{ł}\frac{w_{ww-\mathcal{G}(\text{ȼ})}\left(1+\text{Ƭ}\left(\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}\right)\right)}{\sum_{\text{đ}=1}^{\text{ł}}w_{ww-\text{đ}}\left(1+\text{Ƭ}\left(\mathbb{U}_{BCFS-\text{đ}}\right)\right)}}\right)^{3_{VP-\text{ȼ}}}\right)\right)^{\frac{1}{\text{ł}!}}\right)^{\frac{1}{\sum_{\text{ȼ}=1}^{\text{ł}}3_{VP-\text{ȼ}}}}\\
\left(1-\left(\prod_{\mathcal{G}\in\text{Ƥ}_{\text{ł}}}\left(1-\prod_{\text{ȼ}=1}^{\text{ł}}\left(1-\left(1-\Im^{IP}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}\right)^{\text{ł}\frac{w_{ww-\mathcal{G}(\text{ȼ})}\left(1+\text{Ƭ}\left(\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}\right)\right)}{\sum_{\text{đ}=1}^{\text{ł}}w_{ww-\text{đ}}\left(1+\text{Ƭ}\left(\mathbb{U}_{BCFS-\text{đ}}\right)\right)}}\right)^{3_{VP-\text{ȼ}}}\right)\right)^{\frac{1}{\text{ł}!}}\right)^{\frac{1}{\sum_{\text{ȼ}=1}^{\text{ł}}3_{VP-\text{ȼ}}}},\\
-1+\left(1-\left|-\prod_{\mathcal{G}\in\text{Ƥ}_{\text{ł}}}\left(-1+\prod_{\text{ȼ}=1}^{\text{ł}}\left(1-\left|\Im^{RN}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}\right|^{\text{ł}\frac{w_{ww-\mathcal{G}(\text{ȼ})}\left(1+\text{Ƭ}\left(\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}\right)\right)}{\sum_{\text{đ}=1}^{\text{ł}}w_{ww-\text{đ}}\left(1+\text{Ƭ}\left(\mathbb{U}_{BCFS-\text{đ}}\right)\right)}}\right)^{3_{VP-\text{ȼ}}}\right)\right|^{\frac{1}{\text{ł}!}}\right)^{\frac{1}{\sum_{\text{ȼ}=1}^{\text{ł}}3_{VP-\text{ȼ}}}}\\
-1+\left(1-\left|-\prod_{\mathcal{G}\in\text{Ƥ}_{\text{ł}}}\left(-1+\prod_{\text{ȼ}=1}^{\text{ł}}\left(1-\left|\Im^{IN}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}\right|^{\text{ł}\frac{w_{ww-\mathcal{G}(\text{ȼ})}\left(1+\text{Ƭ}\left(\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}\right)\right)}{\sum_{\text{đ}=1}^{\text{ł}}w_{ww-\text{đ}}\left(1+\text{Ƭ}\left(\mathbb{U}_{BCFS-\text{đ}}\right)\right)}}\right)^{3_{VP-\text{ȼ}}}\right)\right|^{\frac{1}{\text{ł}!}}\right)^{\frac{1}{\sum_{\text{ȼ}=1}^{\text{ł}}3_{VP-\text{ȼ}}}}
\end{array}\right) \quad (14)$$

Proof: Omitted since likewise Theorem (1).

Theorem 6: (Idempotency) Consider $\mathbb{U}_{BCFS-\text{ȼ}}=\left(\Im^{P}_{\mathbb{U}_{BCFS-\text{ȼ}}},\Im^{N}_{\mathbb{U}_{BCFS-\text{ȼ}}}\right)=\left(\Im^{RP}_{\mathbb{U}_{BCFS-\text{ȼ}}}+\iota\,\Im^{IP}_{\mathbb{U}_{BCFS-\text{ȼ}}},\Im^{RN}_{\mathbb{U}_{BCFS-\text{ȼ}}}+\iota\,\Im^{IN}_{\mathbb{U}_{BCFS-\text{ȼ}}}\right),\text{ȼ}=1,2,\dots,\text{ł}$ as a collection of BCFNs and $\mathbb{U}_{BCFS-\text{ȼ}}=\mathbb{U}_{BCFS}\;\forall\,\text{ȼ}$, then

$$BCFWPMM^{3_{VP}}(\mathbb{U}_{BCFS-1},\mathbb{U}_{BCFS-2},\dots,\mathbb{U}_{BCFS-\text{ł}})=\mathbb{U}_{BCFS}$$

Theorem 7: (Monotonicity) Consider $\mathbb{U}_{BCFS-\cancel{e}} = \left(\mathfrak{I}^{P}_{\mathbb{U}_{BCFS-\cancel{e}}}, \mathfrak{I}^{N}_{\mathbb{U}_{BCFS-\cancel{e}}}\right) = \left(\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\cancel{e}}} + \iota\,\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\cancel{e}}}, \mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\cancel{e}}} + \iota\,\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\cancel{e}}}\right)$, and $\mathbb{U}^{\#}_{BCFS-\cancel{e}} = \left(\mathfrak{I}^{P}_{\mathbb{U}^{\#}_{BCFS-\cancel{e}}}, \mathfrak{I}^{N}_{\mathbb{U}^{\#}_{BCFS-\cancel{e}}}\right) = \left(\mathfrak{I}^{RP}_{\mathbb{U}^{\#}_{BCFS-\cancel{e}}} + \iota\,\mathfrak{I}^{IP}_{\mathbb{U}^{\#}_{BCFS-\cancel{e}}}, \mathfrak{I}^{RN}_{\mathbb{U}^{\#}_{BCFS-\cancel{e}}} + \iota\,\mathfrak{I}^{IN}_{\mathbb{U}^{\#}_{BCFS-\cancel{e}}}\right), \cancel{e} = 1, 2, \dots, \ell$ as two collections of BCFNs and $\mathbb{U}_{BCFS-\cancel{e}} \leq \mathbb{U}^{\#}_{BCFS-\cancel{e}}$ that is, $\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\cancel{e}}} \leq \mathfrak{I}^{RP}_{\mathbb{U}^{\#}_{BCFS-\cancel{e}}}$, $\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\cancel{e}}} \leq \mathfrak{I}^{IP}_{\mathbb{U}^{\#}_{BCFS-\cancel{e}}}$, $\mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\cancel{e}}} \leq \mathfrak{I}^{RN}_{\mathbb{U}^{\#}_{BCFS-\cancel{e}}}$ and $\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\cancel{e}}} \leq \mathfrak{I}^{IN}_{\mathbb{U}^{\#}_{BCFS-\cancel{e}}}$, then

$$BCFWPMM^{3_{VP}}(\mathbb{U}_{BCFS-1}, \mathbb{U}_{BCFS-2}, \dots, \mathbb{U}_{BCFS-\ell}) \leq BCFWPMM^{3_{VP}}\left(\mathbb{U}^{\#}_{BCFS-1}, \mathbb{U}^{\#}_{BCFS-2}, \dots, \mathbb{U}^{\#}_{BCFS-\ell}\right)$$

Theorem 8: (Boundedness) Consider $\mathbb{U}_{BCFS-\cancel{e}} = \left(\mathfrak{I}^{P}_{\mathbb{U}_{BCFS-\cancel{e}}}, \mathfrak{I}^{N}_{\mathbb{U}_{BCFS-\cancel{e}}}\right) = \left(\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\cancel{e}}} + \iota\,\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\cancel{e}}}, \mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\cancel{e}}} + \iota\,\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\cancel{e}}}\right), \cancel{e} = 1, 2, \dots, \ell$ as a collection of BCFNs and $\mathbb{U}^{-}_{BCFS} = \left(\min_{\cancel{e}}\left\{\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\cancel{e}}}\right\} + \iota \min_{\cancel{e}}\left\{\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\cancel{e}}}\right\}, \max_{\cancel{e}}\left\{\mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\cancel{e}}}\right\} + \iota \max_{\cancel{e}}\left\{\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\cancel{e}}}\right\}\right)$, and $\mathbb{U}^{+}_{BCFS} = \left(\max_{\cancel{e}}\left\{\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\cancel{e}}}\right\} + \iota \max_{\cancel{e}}\left\{\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\cancel{e}}}\right\}, \min_{\cancel{e}}\left\{\mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\cancel{e}}}\right\} + \iota \min_{\cancel{e}}\left\{\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\cancel{e}}}\right\}\right)$, then

$$\mathbb{U}^{-}_{BCFS} \leq BCFWPMM^{3_{VP}}(\mathbb{U}_{BCFS-1}, \mathbb{U}_{BCFS-2}, \dots, \mathbb{U}_{BCFS-\ell}) \leq \mathbb{U}^{+}_{BCFS}$$

Definition 8: Consider $\mathbb{U}_{BCFS-\cancel{e}} = \left(\mathfrak{I}^{P}_{\mathbb{U}_{BCFS-\cancel{e}}}, \mathfrak{I}^{N}_{\mathbb{U}_{BCFS-\cancel{e}}}\right) = \left(\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\cancel{e}}} + \iota\,\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\cancel{e}}}, \mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\cancel{e}}} + \iota\,\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\cancel{e}}}\right), \cancel{e} = 1, 2, \dots, \ell$ as a collection of BCFNs and $3_{VP} = (3_{VP-1}, 3_{VP-2}, \dots 3_{VP-\ell}) \in \mathfrak{R}^{\ell}$ as a vector of parameters, then

$$BCFPDMM^{3_{VP}}(\mathbb{U}_{BCFS-1}, \mathbb{U}_{BCFS-2}, \dots, \mathbb{U}_{BCFS-\ell})$$
$$= \frac{1}{\sum_{\cancel{e}=1}^{\ell} 3_{VP-\cancel{e}}} \left(\prod_{\mathcal{G} \in P_{\ell}} \sum_{\cancel{e}=1}^{\ell} \left(3_{VP-\cancel{e}} \left(\mathbb{U}_{BCFS-\mathcal{G}(\cancel{e})}\right)^{\frac{\ell\left(1+\mathcal{T}\left(\mathbb{U}_{BCFS-\mathcal{G}(\cancel{e})}\right)\right)}{\sum_{d=1}^{\ell}\left(1+\mathcal{T}(\mathbb{U}_{BCFS-d})\right)}} \right) \right)^{\frac{1}{\ell!}} \quad (15)$$

Would say to be a BCFPDMM operator, where P_{ℓ} would signify the collection of permutations, $\mathcal{G}(\cancel{e})$ would be any permutation of $(1, 2, \dots, \ell)$ and $\mathcal{T}(\mathbb{U}_{BCFS-d}) = \sum_{\substack{d=1 \\ d \neq \cancel{e}}}^{\ell} Supp(\mathbb{U}_{BCFS-\cancel{e}}, \mathbb{U}_{BCFS-d})$, where $Supp(\mathbb{U}_{BCFS-\cancel{e}}, \mathbb{U}_{BCFS-d})$ holds the properties underneath

1. $Supp(\mathbb{U}_{BCFS-\cancel{e}}, \mathbb{U}_{BCFS-d}) \in [0, 1]$
2. $Supp(\mathbb{U}_{BCFS-\cancel{e}}, \mathbb{U}_{BCFS-d}) = Supp(\mathbb{U}_{BCFS-z}, \mathbb{U}_{BCFS-\cancel{e}})$
3. If $\mathfrak{D}_{s}(\mathbb{U}_{BCFS-\cancel{e}}, \mathbb{U}_{BCFS-d}) < \mathfrak{D}_{s}\left(\mathbb{U}_{BCFS-\cancel{e}'}, \mathbb{U}_{BCFS-d'}\right)$, then $Supp(\mathbb{U}_{BCFS-\cancel{e}}, \mathbb{U}_{BCFS-d}) > Supp\left(\mathbb{U}_{BCFS-\cancel{e}'}, \mathbb{U}_{BCFS-d'}\right)$, where $\mathfrak{D}_{s}(\mathbb{U}_{BCFS-\cancel{e}}, \mathbb{U}_{BCFS-d})$ signifies the distance between $\mathbb{U}_{BCFS-\cancel{e}}$ and $\mathbb{U}_{BCFS-d}$.

and $Supp(\mathbb{U}_{BCFS-\cancel{e}}, \mathbb{U}_{BCFS-d})$ is known as the support degree for $\mathbb{U}_{BCFS-\cancel{e}}$ and $\mathbb{U}_{BCFS-d}$.

To simply Eq. (15), we consider $\frac{\left(1+\mathcal{T}\left(\mathbb{U}_{BCFS-\mathcal{G}(\cancel{e})}\right)\right)}{\sum_{d=1}^{\ell}\left(1+\mathcal{T}(\mathbb{U}_{BCFS-d})\right)} = \mathfrak{f}_{\cancel{e}}$, then Eq. (15) is converted to the following form

$$BCFPDMM^{3_{VP}}(\mathbb{U}_{BCFS-1}, \mathbb{U}_{BCFS-2}, \dots, \mathbb{U}_{BCFS-\ell}) = \frac{1}{\sum_{\cancel{e}=1}^{\ell} 3_{VP-\cancel{e}}} \left(\prod_{\mathcal{G} \in P_{\ell}} \sum_{\cancel{e}=1}^{\ell} \left(3_{VP-\cancel{e}} \left(\mathbb{U}_{BCFS-\mathcal{G}(\cancel{e})}\right)^{\ell \mathfrak{f}_{\cancel{e}}} \right) \right)^{\frac{1}{\ell!}} \quad (16)$$

and for appropriateness, we can consider that $(\mathfrak{f}_1, \mathfrak{f}_2, \dots, \mathfrak{f}_{\ell})^T$ would be the power weight vector with the condition that $\mathfrak{f}_{\cancel{e}} \in [0, 1]$ and $\sum_{\cancel{e}=1}^{\ell} \mathfrak{f}_{\cancel{e}} = 1$.

Theorem 9: Consider $\mathbb{U}_{BCFS-\cancel{c}} = \left(\mathfrak{I}^{P}_{\mathbb{U}_{BCFS-\cancel{c}}}, \mathfrak{I}^{N}_{\mathbb{U}_{BCFS-\cancel{c}}}\right) = \left(\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\cancel{c}}} + \iota\, \mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\cancel{c}}}, \mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\cancel{c}}} + \iota\, \mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\cancel{c}}}\right), \cancel{c} = 1, 2, \dots, \ell$ as a collection of BCFNs, then the result after aggregating the BCFNs with the assistance of Eq. (16) is yet again a BCFN

$$BCFPDMM^{\mathfrak{Z}_{VP}}\left(\mathbb{U}_{BCFS-1}, \mathbb{U}_{BCFS-2}, \dots, \mathbb{U}_{BCFS-\ell}\right)$$

$$= \left(\begin{array}{c} 1 - \left(1 - \left(\prod_{\mathcal{G}\in P_\ell}\left(1 - \prod_{\cancel{c}=1}^{\ell}\left(1 - \left(\left(\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}}\right)^{\mathfrak{F}_{\cancel{c}}}\right)^{\mathfrak{Z}_{VP-\cancel{c}}}\right)\right)\right)^{\frac{1}{\ell!}}\right)^{\frac{1}{\sum_{\cancel{c}=1}^{\ell}\mathfrak{Z}_{VP-\cancel{c}}}} \\ +\iota \left(1 - \left(1 - \left(\prod_{\mathcal{G}\in P_\ell}\left(1 - \prod_{\cancel{c}=1}^{\ell}\left(1 - \left(\left(\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}}\right)^{\mathfrak{F}_{\cancel{c}}}\right)^{\mathfrak{Z}_{VP-\cancel{c}}}\right)\right)\right)^{\frac{1}{\ell!}}\right)^{\frac{1}{\sum_{\cancel{c}=1}^{\ell}\mathfrak{Z}_{VP-\cancel{c}}}}\right), \\ -\left|-1 + \left(\prod_{\mathcal{G}\in P_\ell}\left(1 - \prod_{\cancel{c}=1}^{\ell}\left|-1 + \left(1 + \mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}}\right)^{\mathfrak{F}_{\cancel{c}}}\right|^{\mathfrak{Z}_{VP-\cancel{c}}}\right)\right)^{\frac{1}{\ell!}}\right|^{\frac{1}{\sum_{\cancel{c}=1}^{\ell}\mathfrak{Z}_{VP-\cancel{c}}}} \\ +\iota \left(-\left|-1 + \left(\prod_{\mathcal{G}\in P_\ell}\left(1 - \prod_{\cancel{c}=1}^{\ell}\left|-1 + \left(1 + \mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}}\right)^{\mathfrak{F}_{\cancel{c}}}\right|^{\mathfrak{Z}_{VP-\cancel{c}}}\right)\right)^{\frac{1}{\ell!}}\right|^{\frac{1}{\sum_{\cancel{c}=1}^{\ell}\mathfrak{Z}_{VP-\cancel{c}}}}\right) \end{array} \right) \quad (17)$$

Proof: With the help of Def (3), we have

$$\left(\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}\right)^{\mathfrak{F}_{\cancel{c}}} = \left(\begin{array}{c} \left(\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}}\right)^{\mathfrak{F}_{\cancel{c}}} + \iota \left(\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}}\right)^{\mathfrak{F}_{\cancel{c}}}, \\ -1 + \left(1 + \mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}}\right)^{\mathfrak{F}_{\cancel{c}}} + \iota \left(-1 + \left(1 + \mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}}\right)^{\mathfrak{F}_{\cancel{c}}}\right) \end{array} \right)$$

and

$$\mathfrak{Z}_{VP-\cancel{c}}\left(\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}\right)^{\mathfrak{F}_{\cancel{c}}} = \left(\begin{array}{c} 1 - \left(1 - \left(\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}}\right)^{\mathfrak{F}_{\cancel{c}}}\right)^{\mathfrak{Z}_{VP-\cancel{c}}} + \iota \left(1 - \left(1 - \left(\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}}\right)^{\mathfrak{F}_{\cancel{c}}}\right)^{\mathfrak{Z}_{VP-\cancel{c}}}\right), \\ -\left|-1 + \left(1 + \mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}}\right)^{\mathfrak{F}_{\cancel{c}}}\right|^{\mathfrak{Z}_{VP-\cancel{c}}} + \iota \left(-\left|-1 + \left(1 + \mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}}\right)^{\mathfrak{F}_{\cancel{c}}}\right|^{\mathfrak{Z}_{VP-\cancel{c}}}\right) \end{array} \right)$$

then,

$$\sum_{\cancel{c}=1}^{\ell}\left(\mathfrak{Z}_{VP-\cancel{c}}\left(\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}\right)^{\mathfrak{F}_{\cancel{c}}}\right)$$

$$= \left(\begin{array}{c} 1 - \prod_{\cancel{c}=1}^{\ell}\left(1 - \left(\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}}\right)^{\mathfrak{F}_{\cancel{c}}}\right)^{\mathfrak{Z}_{VP-\cancel{c}}} + \iota \left(1 - \prod_{\cancel{c}=1}^{\ell}\left(1 - \left(\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}}\right)^{\mathfrak{F}_{\cancel{c}}}\right)^{\mathfrak{Z}_{VP-\cancel{c}}}\right), \\ -\prod_{\cancel{c}=1}^{\ell}\left|-1 + \left(1 + \mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}}\right)^{\mathfrak{F}_{\cancel{c}}}\right|^{\mathfrak{Z}_{VP-\cancel{c}}} + \iota \left(-\prod_{\cancel{c}=1}^{\ell}\left|-1 + \left(1 + \mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\mathcal{G}(\cancel{c})}}\right)^{\mathfrak{F}_{\cancel{c}}}\right|^{\mathfrak{Z}_{VP-\cancel{c}}}\right) \end{array} \right)$$

next,

$$\prod_{\mathcal{G}\in P_{\text{ł}}}\sum_{\text{ȼ}=1}^{\text{ł}}\left(\text{Ʒ}_{VP-\text{ȼ}}\left(\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}\right)^{\text{łF}_{\text{ȼ}}}\right)=\begin{pmatrix}\prod_{\mathcal{G}\in P_{\text{ł}}}\left(1-\prod_{\text{ȼ}=1}^{\text{ł}}\left(1-\left(\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}\right)^{\text{łF}_{\text{ȼ}}}\right)^{\text{Ʒ}_{VP-\text{ȼ}}}\right)\\ +\iota\prod_{\mathcal{G}\in P_{\text{ł}}}\left(1-\prod_{\text{ȼ}=1}^{\text{ł}}\left(1-\left(\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}\right)^{\text{łF}_{\text{ȼ}}}\right)^{\text{Ʒ}_{VP-\text{ȼ}}}\right),\\ -1+\prod_{\mathcal{G}\in P_{\text{ł}}}\left(1-\prod_{\text{ȼ}=1}^{\text{ł}}\left|-1+\left(1+\mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}\right)^{\text{łF}_{\text{ȼ}}}\right|^{\text{Ʒ}_{VP-\text{ȼ}}}\right)\\ +\iota\left(-1+\prod_{\mathcal{G}\in P_{\text{ł}}}\left(1-\prod_{\text{ȼ}=1}^{\text{ł}}\left|-1+\left(1+\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}\right)^{\text{łF}_{\text{ȼ}}}\right|^{\text{Ʒ}_{VP-\text{ȼ}}}\right)\right)\end{pmatrix}$$

$$\left(\prod_{\mathcal{G}\in P_{\text{ł}}}\sum_{\text{ȼ}=1}^{\text{ł}}\left(\text{Ʒ}_{VP-\text{ȼ}}\left(\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}\right)^{\text{łF}_{\text{ȼ}}}\right)\right)^{\frac{1}{\text{ł}!}}=\begin{pmatrix}\left(\prod_{\mathcal{G}\in P_{\text{ł}}}\left(1-\prod_{\text{ȼ}=1}^{\text{ł}}\left(1-\left(\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}\right)^{\text{łF}_{\text{ȼ}}}\right)^{\text{Ʒ}_{VP-\text{ȼ}}}\right)\right)^{\frac{1}{\text{ł}!}}\\ +\iota\left(\prod_{\mathcal{G}\in P_{\text{ł}}}\left(1-\prod_{\text{ȼ}=1}^{\text{ł}}\left(1-\left(\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}\right)^{\text{łF}_{\text{ȼ}}}\right)^{\text{Ʒ}_{VP-\text{ȼ}}}\right)\right)^{\frac{1}{\text{ł}!}},\\ -1+\left(\prod_{\mathcal{G}\in P_{\text{ł}}}\left(1-\prod_{\text{ȼ}=1}^{\text{ł}}\left|-1+\left(1+\mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}\right)^{\text{łF}_{\text{ȼ}}}\right|^{\text{Ʒ}_{VP-\text{ȼ}}}\right)\right)^{\frac{1}{\text{ł}!}}\\ +\iota\left(-1+\left(\prod_{\mathcal{G}\in P_{\text{ł}}}\left(1-\prod_{\text{ȼ}=1}^{\text{ł}}\left|-1+\left(1+\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}}\right)^{\text{łF}_{\text{ȼ}}}\right|^{\text{Ʒ}_{VP-\text{ȼ}}}\right)\right)^{\frac{1}{\text{ł}!}}\right)\end{pmatrix}$$

Finally,

$$
\frac{1}{\sum_{ȼ=1}^{ł} Ʒ_{VP-ȼ}}\left(\prod_{\mathcal{G}\in P_{ł}}\sum_{ȼ=1}^{ł}\left(Ʒ_{VP-ȼ}\left(\mathbb{U}_{BCFS-\mathcal{G}(ȼ)}\right)^{łƑ_{ȼ}}\right)\right)^{\frac{1}{ł!}}
$$

$$
=\left(\begin{array}{c}
1-\left(1-\left(\prod_{\mathcal{G}\in P_{ł}}\left(1-\prod_{ȼ=1}^{ł}\left(1-\left(\mathfrak{I}_{\mathbb{U}_{BCFS-\mathcal{G}(ȼ)}}^{RP}\right)^{łƑ_{ȼ}}\right)^{Ʒ_{VP-ȼ}}\right)\right)^{\frac{1}{ł!}}\right)^{\frac{1}{\sum_{ȼ=1}^{ł} Ʒ_{VP-ȼ}}} \\
+\iota\left(1-\left(1-\left(\prod_{\mathcal{G}\in P_{ł}}\left(1-\prod_{ȼ=1}^{ł}\left(1-\left(\mathfrak{I}_{\mathbb{U}_{BCFS-\mathcal{G}(ȼ)}}^{IP}\right)^{łƑ_{ȼ}}\right)^{Ʒ_{VP-ȼ}}\right)\right)^{\frac{1}{ł!}}\right)^{\frac{1}{\sum_{ȼ=1}^{ł} Ʒ_{VP-ȼ}}}\right), \\
-\left|-1+\left(\prod_{\mathcal{G}\in P_{ł}}\left(1-\prod_{ȼ=1}^{ł}\left|-1+\left(1+\mathfrak{I}_{\mathbb{U}_{BCFS-\mathcal{G}(ȼ)}}^{RN}\right)^{łƑ_{ȼ}}\right|^{Ʒ_{VP-ȼ}}\right)\right)^{\frac{1}{ł!}}\right|^{\frac{1}{\sum_{ȼ=1}^{ł} Ʒ_{VP-ȼ}}} \\
+\iota\left(-\left|-1+\left(\prod_{\mathcal{G}\in P_{ł}}\left(1-\prod_{ȼ=1}^{ł}\left|-1+\left(1+\mathfrak{I}_{\mathbb{U}_{BCFS-\mathcal{G}(ȼ)}}^{IN}\right)^{łƑ_{ȼ}}\right|^{Ʒ_{VP-ȼ}}\right)\right)^{\frac{1}{ł!}}\right|^{\frac{1}{\sum_{ȼ=1}^{ł} Ʒ_{VP-ȼ}}}\right)
\end{array}\right)
$$

This implies that

$$
BCFPDMM^{Ʒ_{VP}}\left(\mathbb{U}_{BCFS-1},\mathbb{U}_{BCFS-2},\dots,\mathbb{U}_{BCFS-ł}\right)
$$

$$
=\left(\begin{array}{c}
1-\left(1-\left(\prod_{\mathcal{G}\in P_{ł}}\left(1-\prod_{ȼ=1}^{ł}\left(1-\left(\mathfrak{I}_{\mathbb{U}_{BCFS-\mathcal{G}(ȼ)}}^{RP}\right)^{łƑ_{ȼ}}\right)^{Ʒ_{VP-ȼ}}\right)\right)^{\frac{1}{ł!}}\right)^{\frac{1}{\sum_{ȼ=1}^{ł} Ʒ_{VP-ȼ}}} \\
+\iota\left(1-\left(1-\left(\prod_{\mathcal{G}\in P_{ł}}\left(1-\prod_{ȼ=1}^{ł}\left(1-\left(\mathfrak{I}_{\mathbb{U}_{BCFS-\mathcal{G}(ȼ)}}^{IP}\right)^{łƑ_{ȼ}}\right)^{Ʒ_{VP-ȼ}}\right)\right)^{\frac{1}{ł!}}\right)^{\frac{1}{\sum_{ȼ=1}^{ł} Ʒ_{VP-ȼ}}}\right), \\
-\left|-1+\left(\prod_{\mathcal{G}\in P_{ł}}\left(1-\prod_{ȼ=1}^{ł}\left|-1+\left(1+\mathfrak{I}_{\mathbb{U}_{BCFS-\mathcal{G}(ȼ)}}^{RN}\right)^{łƑ_{ȼ}}\right|^{Ʒ_{VP-ȼ}}\right)\right)^{\frac{1}{ł!}}\right|^{\frac{1}{\sum_{ȼ=1}^{ł} Ʒ_{VP-ȼ}}} \\
+\iota\left(-\left|-1+\left(\prod_{\mathcal{G}\in P_{ł}}\left(1-\prod_{ȼ=1}^{ł}\left|-1+\left(1+\mathfrak{I}_{\mathbb{U}_{BCFS-\mathcal{G}(ȼ)}}^{IN}\right)^{łƑ_{ȼ}}\right|^{Ʒ_{VP-ȼ}}\right)\right)^{\frac{1}{ł!}}\right|^{\frac{1}{\sum_{ȼ=1}^{ł} Ʒ_{VP-ȼ}}}\right)
\end{array}\right)
$$

Theorem 10: (Idempotency) Consider $\mathbb{U}_{BCFS-ȼ}=\left(\mathfrak{I}_{\mathbb{U}_{BCFS-ȼ}}^{P},\mathfrak{I}_{\mathbb{U}_{BCFS-ȼ}}^{N}\right)=\left(\mathfrak{I}_{\mathbb{U}_{BCFS-ȼ}}^{RP}+\iota\,\mathfrak{I}_{\mathbb{U}_{BCFS-ȼ}}^{IP},\mathfrak{I}_{\mathbb{U}_{BCFS-ȼ}}^{RN}+\iota\,\mathfrak{I}_{\mathbb{U}_{BCFS-ȼ}}^{IN}\right), ȼ=1,2,\dots,ł$ as a collection of BCFNs and $\mathbb{U}_{BCFS-ȼ}=\mathbb{U}_{BCFS}\ \forall\ ȼ$, then

$$
BCFPDMM^{Ʒ_{VP}}\left(\mathbb{U}_{BCFS-1},\mathbb{U}_{BCFS-2},\dots,\mathbb{U}_{BCFS-ł}\right)=\mathbb{U}_{BCFS}
$$

Theorem 11: (Monotonicity) Consider $\mathbb{U}_{BCFS-\text{ȼ}} = \left(\mathfrak{I}^{P}_{\mathbb{U}_{BCFS-\text{ȼ}}}, \mathfrak{I}^{N}_{\mathbb{U}_{BCFS-\text{ȼ}}}\right) = \left(\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\text{ȼ}}} + \iota\,\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\text{ȼ}}}, \mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\text{ȼ}}} + \iota\,\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\text{ȼ}}}\right)$, and $\mathbb{U}^{\#}_{BCFS-\text{ȼ}} = \left(\mathfrak{I}^{P}_{\mathbb{U}^{\#}_{BCFS-\text{ȼ}}}, \mathfrak{I}^{N}_{\mathbb{U}^{\#}_{BCFS-\text{ȼ}}}\right) = \left(\mathfrak{I}^{RP}_{\mathbb{U}^{\#}_{BCFS-\text{ȼ}}} + \iota\,\mathfrak{I}^{IP}_{\mathbb{U}^{\#}_{BCFS-\text{ȼ}}}, \mathfrak{I}^{RN}_{\mathbb{U}^{\#}_{BCFS-\text{ȼ}}} + \iota\,\mathfrak{I}^{IN}_{\mathbb{U}^{\#}_{BCFS-\text{ȼ}}}\right), \text{ȼ} = 1, 2, \dots, \text{ƚ}$ as two collections of BCFNs and $\mathbb{U}_{BCFS-\text{ȼ}} \leq \mathbb{U}^{\#}_{BCFS-\text{ȼ}}$ that is, $\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\text{ȼ}}} \leq \mathfrak{I}^{RP}_{\mathbb{U}^{\#}_{BCFS-\text{ȼ}}}$, $\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\text{ȼ}}} \leq \mathfrak{I}^{IP}_{\mathbb{U}^{\#}_{BCFS-\text{ȼ}}}$, $\mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\text{ȼ}}} \leq \mathfrak{I}^{RN}_{\mathbb{U}^{\#}_{BCFS-\text{ȼ}}}$ and $\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\text{ȼ}}} \leq \mathfrak{I}^{IN}_{\mathbb{U}^{\#}_{BCFS-\text{ȼ}}}$, then

$$BCFPDMM^{\mathfrak{z}_{VP}}(\mathbb{U}_{BCFS-1}, \mathbb{U}_{BCFS-2}, \dots, \mathbb{U}_{BCFS-\text{ƚ}}) \leq BCFPDMM^{\mathfrak{z}_{VP}}\left(\mathbb{U}^{\#}_{BCFS-1}, \mathbb{U}^{\#}_{BCFS-2}, \dots, \mathbb{U}^{\#}_{BCFS-\text{ƚ}}\right)$$

Theorem 12: (Boundedness) Consider $\mathbb{U}_{BCFS-\text{ȼ}} = \left(\mathfrak{I}^{P}_{\mathbb{U}_{BCFS-\text{ȼ}}}, \mathfrak{I}^{N}_{\mathbb{U}_{BCFS-\text{ȼ}}}\right) = \left(\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\text{ȼ}}} + \iota\,\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\text{ȼ}}}, \mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\text{ȼ}}} + \iota\,\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\text{ȼ}}}\right), \text{ȼ} = 1, 2, \dots, \text{ƚ}$ as a collection of BCFNs and $\mathbb{U}^{-}_{BCFS} = \left(\min_{\text{ȼ}}\left\{\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\text{ȼ}}}\right\} + \iota \min_{\text{ȼ}}\left\{\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\text{ȼ}}}\right\}, \max_{\text{ȼ}}\left\{\mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\text{ȼ}}}\right\} + \iota \max_{\text{ȼ}}\left\{\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\text{ȼ}}}\right\}\right)$, and $\mathbb{U}^{+}_{BCFS} = \left(\max_{\text{ȼ}}\left\{\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\text{ȼ}}}\right\} + \iota \max_{\text{ȼ}}\left\{\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\text{ȼ}}}\right\}, \min_{\text{ȼ}}\left\{\mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\text{ȼ}}}\right\} + \iota \min_{\text{ȼ}}\left\{\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\text{ȼ}}}\right\}\right)$, then

$$\mathbb{U}^{-}_{BCFS} \leq BCFPDMM^{\mathfrak{z}_{VP}}(\mathbb{U}_{BCFS-1}, \mathbb{U}_{BCFS-2}, \dots, \mathbb{U}_{BCFS-\text{ƚ}}) \leq \mathbb{U}^{+}_{BCFS}$$

Definition 9: Consider $\mathbb{U}_{BCFS-\text{ȼ}} = \left(\mathfrak{I}^{P}_{\mathbb{U}_{BCFS-\text{ȼ}}}, \mathfrak{I}^{N}_{\mathbb{U}_{BCFS-\text{ȼ}}}\right) = \left(\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\text{ȼ}}} + \iota\,\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\text{ȼ}}}, \mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\text{ȼ}}} + \iota\,\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\text{ȼ}}}\right), \text{ȼ} = 1, 2, \dots, \text{ƚ}$ as a collection of BCFNs and $\mathfrak{z}_{VP} = (\mathfrak{z}_{VP-1}, \mathfrak{z}_{VP-2}, \dots \mathfrak{z}_{VP-\text{ƚ}}) \in \mathfrak{R}^{\text{ƚ}}$ as a vector of parameters, then

$$BCFWPDMM^{\mathfrak{z}_{VP}}(\mathbb{U}_{BCFS-1}, \mathbb{U}_{BCFS-2}, \dots, \mathbb{U}_{BCFS-\text{ƚ}})$$
$$= \frac{1}{\sum_{\text{ȼ}=1}^{\text{ƚ}} \mathfrak{z}_{VP-\text{ȼ}}} \left(\prod_{\mathcal{G} \in P_{\text{ƚ}}} \sum_{\text{ȼ}=1}^{\text{ƚ}} \left(\mathfrak{z}_{VP-\text{ȼ}} \left(\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}\right)^{\text{ƚ} \frac{w_{wv-\mathcal{G}(\text{ȼ})}\left(1+\mathcal{T}\left(\mathbb{U}_{BCFS-\mathcal{G}(\text{ȼ})}\right)\right)}{\sum_{\text{ɗ}=1}^{\text{ƚ}} w_{wv-\text{ɗ}}\left(1+\mathcal{T}\left(\mathbb{U}_{BCFS-\text{ɗ}}\right)\right)}} \right) \right)^{\frac{1}{\text{ƚ}!}} \quad (18)$$

Would say to be a BCFWPDMM operator, where $P_{\text{ƚ}}$ would signify the collection of permutations, $\mathcal{G}(\text{ȼ})$ would be any permutation of $(1, 2, \dots, \text{ƚ})$, $\mathcal{T}(\mathbb{U}_{BCFS-\text{ɗ}}) = \sum_{\substack{\text{ɗ}=1 \\ \text{ɗ}\neq\text{ȼ}}}^{\text{ƚ}} Supp(\mathbb{U}_{BCFS-\text{ȼ}}, \mathbb{U}_{BCFS-\text{ɗ}})$, $Supp(\mathbb{U}_{BCFS-\text{ȼ}}, \mathbb{U}_{BCFS-\text{ɗ}})$ is known as the support degree for $\mathbb{U}_{BCFS-\text{ȼ}}$ and $\mathbb{U}_{BCFS-\text{ɗ}}$ and $w_{wv} = (w_{wv-1}, w_{wv-2}, \dots, w_{wv-\text{ƚ}})$ would be the weight vector meet with condition that $\forall\, \text{ƚ}\; 0 \leq w_{wv-\text{ȼ}} \leq 1$ and $\sum_{\text{ȼ}=1}^{\text{ƚ}} w_{wv-\text{ȼ}} = 1$.

Theorem 13: Consider $\mathbb{U}_{BCFS-\text{ȼ}} = \left(\mathfrak{I}^{P}_{\mathbb{U}_{BCFS-\text{ȼ}}}, \mathfrak{I}^{N}_{\mathbb{U}_{BCFS-\text{ȼ}}}\right) = \left(\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\text{ȼ}}} + \iota\,\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\text{ȼ}}}, \mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\text{ȼ}}} + \iota\,\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\text{ȼ}}}\right), \text{ȼ} = 1, 2, \dots, \text{ƚ}$ as a collection of BCFNs, then the result after aggregating the BCFNs with the assistance of Eq. (18) is yet again a BCFN

$BCFWPDMM^{3_{VP}}(\mathbb{U}_{BCFS-1}, \mathbb{U}_{BCFS-2}, \dots, \mathbb{U}_{BCFS-ł})$

$$= \left(\begin{array}{l} 1 - \left(1 - \left(\prod_{\mathcal{G}\in P_{ł}}\left(1 - \prod_{ȼ=1}^{ł}\left(1 - \left(\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-\mathcal{G}(ȼ)}}\right)^{ł\frac{w_{wv-\mathcal{G}(ȼ)}\left(1+\mathcal{T}\left(\mathbb{U}_{BCFS-\mathcal{G}(ȼ)}\right)\right)}{\sum_{ɗ=1}^{ł} w_{wv-ɗ}\left(1+\mathcal{T}\left(\mathbb{U}_{BCFS-ɗ}\right)\right)}}\right)^{3_{VP-ȼ}}\right)\right)^{\frac{1}{ł!}}\right)^{\frac{1}{\sum_{ȼ=1}^{ł} 3_{VP-ȼ}}} \\ +\iota\left(1 - \left(1 - \left(\prod_{\mathcal{G}\in P_{ł}}\left(1 - \prod_{ȼ=1}^{ł}\left(1 - \left(\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-\mathcal{G}(ȼ)}}\right)^{ł\frac{w_{wv-\mathcal{G}(ȼ)}\left(1+\mathcal{T}\left(\mathbb{U}_{BCFS-\mathcal{G}(ȼ)}\right)\right)}{\sum_{ɗ=1}^{ł} w_{wv-ɗ}\left(1+\mathcal{T}\left(\mathbb{U}_{BCFS-ɗ}\right)\right)}}\right)^{3_{VP-ȼ}}\right)\right)^{\frac{1}{ł!}}\right)^{\frac{1}{\sum_{ȼ=1}^{ł} 3_{VP-ȼ}}}\right), \\ -\left|-1 + \left(\prod_{\mathcal{G}\in P_{ł}}\left(1 - \prod_{ȼ=1}^{ł}\left|-1 + \left(1 + \mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-\mathcal{G}(ȼ)}}\right)^{ł\frac{w_{wv-\mathcal{G}(ȼ)}\left(1+\mathcal{T}\left(\mathbb{U}_{BCFS-\mathcal{G}(ȼ)}\right)\right)}{\sum_{ɗ=1}^{ł} w_{wv-ɗ}\left(1+\mathcal{T}\left(\mathbb{U}_{BCFS-ɗ}\right)\right)}}\right|^{3_{VP-ȼ}}\right)\right)^{\frac{1}{ł!}}\right|^{\frac{1}{\sum_{ȼ=1}^{ł} 3_{VP-ȼ}}} \\ +\iota\left(-\left|-1 + \left(\prod_{\mathcal{G}\in P_{ł}}\left(1 - \prod_{ȼ=1}^{ł}\left|-1 + \left(1 + \mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-\mathcal{G}(ȼ)}}\right)^{ł\frac{w_{wv-\mathcal{G}(ȼ)}\left(1+\mathcal{T}\left(\mathbb{U}_{BCFS-\mathcal{G}(ȼ)}\right)\right)}{\sum_{ɗ=1}^{ł} w_{wv-ɗ}\left(1+\mathcal{T}\left(\mathbb{U}_{BCFS-ɗ}\right)\right)}}\right|^{3_{VP-ȼ}}\right)\right)^{\frac{1}{ł!}}\right|^{\frac{1}{\sum_{ȼ=1}^{ł} 3_{VP-ȼ}}}\right) \end{array} \right) \quad (19)$$

Proof: Omitted since likewise to Theorem (9).

Theorem 14: (Idempotency) Consider $\mathbb{U}_{BCFS-ȼ} = \left(\mathfrak{I}^{P}_{\mathbb{U}_{BCFS-ȼ}}, \mathfrak{I}^{N}_{\mathbb{U}_{BCFS-ȼ}}\right) = \left(\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-ȼ}} + \iota\,\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-ȼ}}, \mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-ȼ}} + \iota\,\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-ȼ}}\right), ȼ = 1, 2, \dots, ł$ as a collection of BCFNs and $\mathbb{U}_{BCFS-ȼ} = \mathbb{U}_{BCFS}\ \forall\, ȼ$, then

$$BCFWPDMM^{3_{VP}}(\mathbb{U}_{BCFS-1}, \mathbb{U}_{BCFS-2}, \dots, \mathbb{U}_{BCFS-ł}) = \mathbb{U}_{BCFS}$$

Theorem 15: (Monotonicity) Consider $\mathbb{U}_{BCFS-ȼ} = \left(\mathfrak{I}^{P}_{\mathbb{U}_{BCFS-ȼ}}, \mathfrak{I}^{N}_{\mathbb{U}_{BCFS-ȼ}}\right) = \left(\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-ȼ}} + \iota\,\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-ȼ}}, \mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-ȼ}} + \iota\,\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-ȼ}}\right)$, and $\mathbb{U}^{\#}_{BCFS-ȼ} = \left(\mathfrak{I}^{P}_{\mathbb{U}^{\#}_{BCFS-ȼ}}, \mathfrak{I}^{N}_{\mathbb{U}^{\#}_{BCFS-ȼ}}\right) = \left(\mathfrak{I}^{RP}_{\mathbb{U}^{\#}_{BCFS-ȼ}} + \iota\,\mathfrak{I}^{IP}_{\mathbb{U}^{\#}_{BCFS-ȼ}}, \mathfrak{I}^{RN}_{\mathbb{U}^{\#}_{BCFS-ȼ}} + \iota\,\mathfrak{I}^{IN}_{\mathbb{U}^{\#}_{BCFS-ȼ}}\right), ȼ = 1, 2, \dots, ł$ as two collections of BCFNs and $\mathbb{U}_{BCFS-ȼ} \le \mathbb{U}^{\#}_{BCFS-ȼ}$ that is, $\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-ȼ}} \le \mathfrak{I}^{RP}_{\mathbb{U}^{\#}_{BCFS-ȼ}}$, $\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-ȼ}} \le \mathfrak{I}^{IP}_{\mathbb{U}^{\#}_{BCFS-ȼ}}$, $\mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-ȼ}} \le \mathfrak{I}^{RN}_{\mathbb{U}^{\#}_{BCFS-ȼ}}$ and $\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-ȼ}} \le \mathfrak{I}^{IN}_{\mathbb{U}^{\#}_{BCFS-ȼ}}$, then

$$BCFWPDMM^{3_{VP}}(\mathbb{U}_{BCFS-1}, \mathbb{U}_{BCFS-2}, \dots, \mathbb{U}_{BCFS-ł}) \le BCFWPDMM^{3_{VP}}\left(\mathbb{U}^{\#}_{BCFS-1}, \mathbb{U}^{\#}_{BCFS-2}, \dots, \mathbb{U}^{\#}_{BCFS-ł}\right)$$

Theorem 16: (Boundedness) Consider $\mathbb{U}_{BCFS-ȼ} = \left(\mathfrak{I}^{P}_{\mathbb{U}_{BCFS-ȼ}}, \mathfrak{I}^{N}_{\mathbb{U}_{BCFS-ȼ}}\right) = \left(\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-ȼ}} + \iota\,\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-ȼ}}, \mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-ȼ}} + \iota\,\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-ȼ}}\right), ȼ = 1, 2, \dots, ł$ as a collection of BCFNs and $\mathbb{U}^{-}_{BCFS} = \left(\min_{ȼ}\left\{\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-ȼ}}\right\} + \iota \min_{ȼ}\left\{\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-ȼ}}\right\}, \max_{ȼ}\left\{\mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-ȼ}}\right\} + \iota \max_{ȼ}\left\{\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-ȼ}}\right\}\right)$, and $\mathbb{U}^{+}_{BCFS} = \left(\max_{ȼ}\left\{\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS-ȼ}}\right\} + \iota \max_{ȼ}\left\{\mathfrak{I}^{IP}_{\mathbb{U}_{BCFS-ȼ}}\right\}, \min_{ȼ}\left\{\mathfrak{I}^{RN}_{\mathbb{U}_{BCFS-ȼ}}\right\} + \iota \min_{ȼ}\left\{\mathfrak{I}^{IN}_{\mathbb{U}_{BCFS-ȼ}}\right\}\right)$, then

$$\mathbb{U}^{-}_{BCFS} \le BCFWPDMM^{3_{VP}}(\mathbb{U}_{BCFS-1}, \mathbb{U}_{BCFS-2}, \dots, \mathbb{U}_{BCFS-ł}) \le \mathbb{U}^{+}_{BCFS}$$

4. | MADM WITHIN BCFS

Consider $\mathfrak{l}$ alternatives $\mathbb{U}_{\mathfrak{A}\mathfrak{L}-1}, \mathbb{U}_{\mathfrak{A}\mathfrak{L}-2}, \ldots, \mathbb{U}_{\mathfrak{A}\mathfrak{L}-\mathfrak{l}}$ and k attributes $E_{at-1}, E_{at-2}, \ldots, E_{at-k}$ with their weight vector $w_{\mathcal{W}\mathcal{V}} = (w_{\mathcal{W}\mathcal{V}-1}, w_{\mathcal{W}\mathcal{V}-2}, \ldots, w_{\mathcal{W}\mathcal{V}-k})$ meet the condition that $\forall\, k\; 0 \leq w_{\mathcal{W}\mathcal{V}-\mathrm{d}} \leq 1$ and $\sum_{\mathrm{d}=1}^{k} w_{\mathcal{W}\mathcal{V}-\mathrm{d}} = 1$. After assessing the alternatives $\mathbb{U}_{\mathfrak{A}\mathfrak{L}-\not{c}}, \not{c} = 1, 2, 3, 4$ with respect to the attributes $E_{at-\mathrm{d}}, \mathrm{d} = 1, 2, 3, 4$, the expert would describe the assessing values in the model of BCFN $\left(\mathfrak{I}^{P}_{\mathbb{U}_{BCFS}}, \mathfrak{I}^{N}_{\mathbb{U}_{BCFS}}\right) = \left(\mathfrak{I}^{RP}_{\mathbb{U}_{BCFS}} + \iota\, \mathfrak{I}^{IP}_{\mathbb{U}_{BCFS}}, \mathfrak{I}^{RN}_{\mathbb{U}_{BCFS}} + \iota\, \mathfrak{I}^{IN}_{\mathbb{U}_{BCFS}}\right)$ and create a decision matrix $\mathbb{D}_{\mathcal{DM}}$. For solving the decision matrix, we have the following stages

Stage 1: The decision matrix should be standardized because usually there are two sorts of attributes, that is, benefit and cost. Thus, the following are the formulas to standardize the matrix.

$$\mathbb{D}^{\mathcal{N}}_{\mathcal{DM}} = \begin{cases} \mathfrak{I}^{RP}_{\mathbb{U}_{BCFS}} + \iota\, \mathfrak{I}^{IP}_{\mathbb{U}_{BCFS}}, \mathfrak{I}^{RN}_{\mathbb{U}_{BCFS}} + \iota\, \mathfrak{I}^{IN}_{\mathbb{U}_{BCFS}} & for\ benefit\ attribute \\ 1 - \mathfrak{I}^{RP}_{\mathbb{U}_{BCFS}} + \iota\left(1 - \mathfrak{I}^{IP}_{\mathbb{U}_{BCFS}}\right), -1 - \mathfrak{I}^{RN}_{\mathbb{U}_{BCFS}} + \iota\left(-1 - \mathfrak{I}^{IN}_{\mathbb{U}_{BCFS}}\right) & for\ \not{c}ost\ attribute \end{cases}$$

Stage 2: Aggregate the standardized matrix by engaging any of the originating operators, which are BCFPMM, BCFWPMM, BCFPDMM, and BCFWPDMM operators.

Stage 3: Engage Eq. (2) to have score values, and in case of equal score values, engage Eq. (3) to have accuracy values.

Stage 4: Engage Stage 3 to have the ranking between all alternatives.

Stage 5: Finish.

4.1. | Case study

XYZ Corporation is a fast-growing mid-sized manufacturing firm operating in the automotive parts industry and is experiencing some issues with the management of its growth in terms of departments and geographical locations. Currently, the company uses separate systems for finance, human resources, inventory, and customer relationship management that are insufficient to meet the company's needs. In the case of XYZ, the management has realized that they need to integrate their various functions and improve the quality of information they use to make decisions; hence the need to adopt an ERP system. The company has selected a recruitment committee from members of the management team, including the IT department, finance department, operations department, and the human resources department. The role of this team is to assess and determine the best ERP system that will fit the company's strategic plan and the needs of the organization. It is the task of the selection committee to decide on the most suitable ERP system from four options, which are listed in Table 1.

Table 1. The ERPs and their explanation.

Symbol	ERPs	Explanation
$\mathbb{U}_{\boldsymbol{\mathfrak{AL}}-1}$	SAP S/4HANA	An all-in-one cloud software that is famous for its strong characteristics and adaptability. SAP S/4HANA has features like real-time data processing, integration of machine learning, and a large number of predefined solutions for various industries. It supports operational flows typically associated with manufacturing industries such as production scheduling, quality control, and supply chain management. The system uses in-memory database technology, which makes the processing and reporting of data very fast.
$\mathbb{U}_{\boldsymbol{\mathfrak{AL}}-2}$	Oracle NetSuite	A modern solution for ERP implemented in the cloud, which is preferred by mid-sized companies for its openness. Oracle NetSuite has financials, inventory, order processing, and customer relationship management in its single application. It offers excellent support for multi-subsidiary and multi-currency environments, which is appropriate for organizations that work globally. NetSuite has SuiteSuccess, which is an implementation approach designed to speed up the implementation process and the time to value.
$\mathbb{U}_{\boldsymbol{\mathfrak{AL}}-3}$	Microsoft Dynamics 365	A mid-market ERP with a focus on CRM and a solid connection to the Microsoft environment. Microsoft Dynamics 365 has a more flexible approach to implementation as it is based on modules, and a business can begin its implementation with the financials and then add other modules as the business expands. It offers strong production functions such as production floor management and product configuration. Microsoft integration with other tools like Office 365 and Power BI improves teamwork and business intelligence, respectively.
$\mathbb{U}_{\boldsymbol{\mathfrak{AL}}-4}$	Infor CloudSuite Industrial	A manufacturing-specific ERP system with strong emphasis on usability and data analysis that is designed for manufacturing industries. Below are some of the features that Infor CloudSuite Industrial brings to automotive suppliers: APS, Quality management, EDI. It offers a contemporary look and feel to the shop floor and gives solid mobile support. AI integration is also provided for decision support and future trends analysis for various aspects of the business.

Based on four critical attributes, which are presented in Table 2.

Table 2. The attributes of the considered ERPs

Symbol	Attributes	Explanation
	Total Cost of Ownership (TCO)	This attribute looks at the first-time costs, costs of licensing, recurrent costs, and other hidden costs within a period of 5 years. It is important in budgeting as well as in ROI analysis.
	Functionality and Feature Set	This attribute assesses the level of satisfaction of XYZ's functional needs in different departments with the different ERP systems. It encompasses evaluation of modules concerning finance, human resources, inventory, production schedule, quality assurance, and customer relationship management with specific reference to automotive parts production.
	Scalability and Flexibility	This attribute evaluates the capability of the ERP system to expand as the business of XYZ expands and to also evolve with the changing needs of the business. Some of them include the ability to incorporate new users, modules, or manufacturing centers, flexibility in the configuration of the workflow, and the reporting structure of the system to suit the nature of the industry.
	Implementation Time and Complexity	This attribute assesses the period and other resources that are expected to be used to accomplish the implementation of the ERP system. They include factors like the level of data migration, user training, and the disruption that the implementation process may have on continuous manufacturing processes.

This decision will have a great bearing on the future performance, growth rate, and market position of XYZ in the automotive parts manufacturing business. The selection committee has to compare and analyze these alternatives in terms of the given attributes and their relevance to XYZ's requirements and goals in the automotive parts manufacturing industry. They provide the weight to each attribute as 0.12, 0.24, 0.36, and 0.28, and their assessment values are in Table 3.

Table 3. The assessment arguments described by the team of experts.

	$\boldsymbol{E_{at-1}}$	$\boldsymbol{E_{at-4}}$	$\boldsymbol{E_{at-3}}$	$\boldsymbol{E_{at-4}}$
$\mathbb{U}_{\mathfrak{AL}-1}$	$\begin{pmatrix}0.67+\iota\,0.47,\\ -0.7-\iota\,0.54\end{pmatrix}$	$\begin{pmatrix}0.89+\iota\,0.96,\\ -0.38-\iota\,0.33\end{pmatrix}$	$\begin{pmatrix}0.51+\iota\,0.49,\\ -0.28-\iota\,0.7\end{pmatrix}$	$\begin{pmatrix}0.76+\iota\,0.46,\\ -0.54-\iota\,0.63\end{pmatrix}$
$\mathbb{U}_{\mathfrak{AL}-2}$	$\begin{pmatrix}0.68+\iota\,0.46,\\ -0.69-\iota\,0.55\end{pmatrix}$	$\begin{pmatrix}0.84+\iota\,0.19,\\ -0.18-\iota\,0.97\end{pmatrix}$	$\begin{pmatrix}0.53+\iota\,0.63,\\ -0.17-\iota\,0.48\end{pmatrix}$	$\begin{pmatrix}0.37+\iota\,0.16,\\ -0.76-\iota\,0.56\end{pmatrix}$
$\mathbb{U}_{\mathfrak{AL}-3}$	$\begin{pmatrix}0.69+\iota\,0.45,\\ -0.68-\iota\,0.56\end{pmatrix}$	$\begin{pmatrix}0.29+\iota\,0.39,\\ -0.39-\iota\,0.54\end{pmatrix}$	$\begin{pmatrix}0.37+\iota\,0.76,\\ -0.39-\iota\,0.29\end{pmatrix}$	$\begin{pmatrix}0.47+\iota\,0.66,\\ -0.5-\iota\,0.68\end{pmatrix}$
$\mathbb{U}_{\mathfrak{AL}-4}$	$\begin{pmatrix}0.7+\iota\,0.44,\\ -0.67-\iota\,0.57\end{pmatrix}$	$\begin{pmatrix}0.69+\iota\,0.94,\\ -0.4-\iota\,0.39\end{pmatrix}$	$\begin{pmatrix}0.47+\iota\,0.8,\\ -0.61-\iota\,0.5\end{pmatrix}$	$\begin{pmatrix}0.22+\iota\,0.87,\\ -0.6-\iota\,0.4\end{pmatrix}$

Due to these complexities, the selected MADM method will have to incorporate quantitative and qualitative aspects of the problem and possible correlations between the attributes to identify the best ERP system for XYZ Corporation.

Stage 1: As we are considering the benefits of attributes, we can skip stage 1.

Stage 2: Aggregated outcomes of the values of Table 3 are explored in Table 4. Where $\mathfrak{Z}_{VP} = (3,3,3,3)$ in the original operators.

Table 4. Aggregated outcomes of the values of Table 3.

Operators	$\mathbb{U}_{\mathfrak{AL}-1}$	$\mathbb{U}_{\mathfrak{AL}-2}$	$\mathbb{U}_{\mathfrak{AL}-3}$	$\mathbb{U}_{\mathfrak{AL}-4}$
BCFPMM	$\begin{pmatrix}0.693 + \iota\, 0.565,\\ -0.470 - \iota\, 0.558\end{pmatrix}$	$\begin{pmatrix}0.579 + \iota\, 0.307,\\ -0.497 - \iota\, 0.758\end{pmatrix}$	$\begin{pmatrix}0.432 + \iota\, 0.544,\\ -0.478 - \iota\, 0.523\end{pmatrix}$	$\begin{pmatrix}0.473 + \iota\, 0.732,\\ -0.558 - \iota\, 0.454\end{pmatrix}$
BCFWPMM	$\begin{pmatrix}0.704 + \iota\, 0.568,\\ -0.469 - \iota\, 0.544\end{pmatrix}$	$\begin{pmatrix}0.584 + \iota\, 0.301,\\ -0.501 - \iota\, 0.755\end{pmatrix}$	$\begin{pmatrix}0.439 + \iota\, 0.531,\\ -0.471 - \iota\, 0.525\end{pmatrix}$	$\begin{pmatrix}0.478 + \iota\, 0.718,\\ -0.545 - \iota\, 0.444\end{pmatrix}$
BCFPDMM	$\begin{pmatrix}0.792 + \iota\, 0.775,\\ -0.448 - \iota\, 0.529\end{pmatrix}$	$\begin{pmatrix}0.715 + \iota\, 0.508,\\ -0.356 - \iota\, 0.616\end{pmatrix}$	$\begin{pmatrix}0.577 + \iota\, 0.669,\\ -0.477 - \iota\, 0.494\end{pmatrix}$	$\begin{pmatrix}0.64 + \iota\, 0.86,\\ -0.56 - \iota\, 0.459\end{pmatrix}$
BCFWPDMM	$\begin{pmatrix}0.794 + \iota\, 0.776,\\ -0.46 - \iota\, 0.522\end{pmatrix}$	$\begin{pmatrix}0.716 + \iota\, 0.498,\\ -0.368 - \iota\, 0.623\end{pmatrix}$	$\begin{pmatrix}0.583 + \iota\, 0.663,\\ -0.485 - \iota\, 0.505\end{pmatrix}$	$\begin{pmatrix}0.643 + \iota\, 0.858,\\ -0.56 - \iota\, 0.46\end{pmatrix}$

Stage 3: Engaged Eq. (2) to have score values of various types of ERPs, which are demonstrated in Table 5.

Table 5. The score values of the arguments are demonstrated in Table 4.

Operators	$\mathcal{S}(\mathbb{U}_{\mathfrak{AL}-1})$	$\mathcal{S}(\mathbb{U}_{\mathfrak{AL}-2})$	$\mathcal{S}(\mathbb{U}_{\mathfrak{AL}-3})$	$\mathcal{S}(\mathbb{U}_{\mathfrak{AL}-4})$
BCFPMM	0.55749	0.4078	0.4978	0.54847
BCFWPMM	0.56466	0.40703	0.4935	0.55187
BCFPDMM	064756	0.56278	0.56874	0.62041
BCFWPDMM	0.64727	0.55579	0.56402	0.62022

Stage 4: By employing the BCFPMM, BCFWPMM, BCFPDMM, and BCFWPDMM operators, we have $\mathbb{U}_{\mathfrak{AL}-1} > \mathbb{U}_{\mathfrak{AL}-4} > \mathbb{U}_{\mathfrak{AL}-3} > \mathbb{U}_{\mathfrak{AL}-2}$ as ranking ordered by keeping in view the score values described in Table 5 and Fig 4. Hence $\mathbb{U}_{\mathfrak{AL}-1}$ is the most superb and $\mathbb{U}_{\mathfrak{AL}-2}$ is the worst alternative among all given alternatives.

Stage 5: Finish.

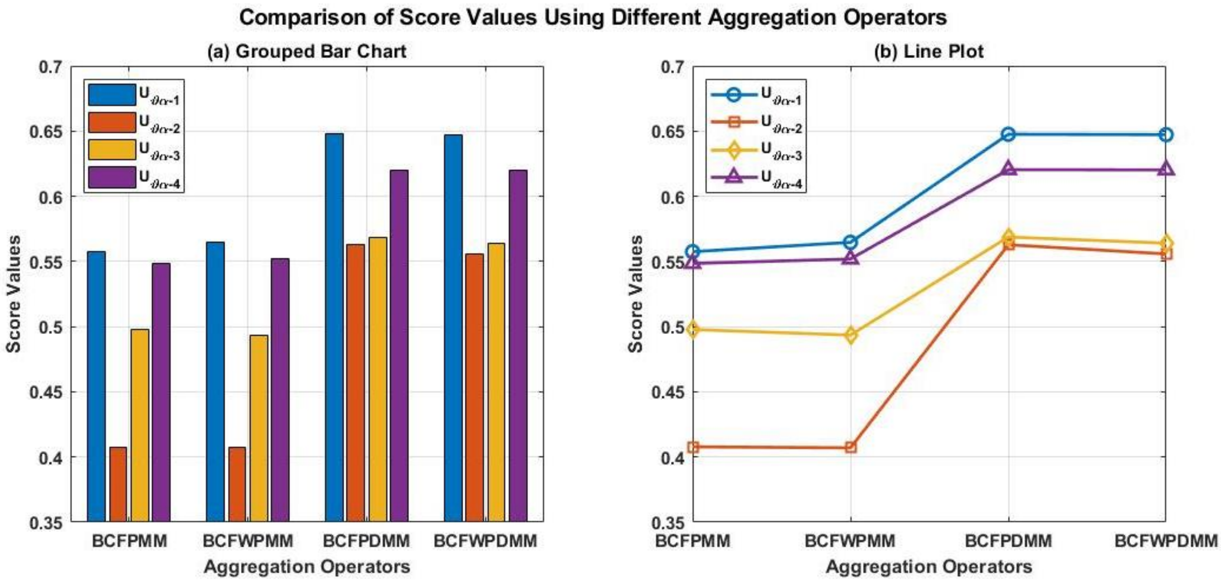

Figure 4. The graphical representation of ranking

5. | COMPARATIVE ANALYSIS

This Section contains the comparative study among the original PMM operators for BCFNs and a few of the prevailing operators for BCFNs and other theories.

Assume the information in Table 1, which is in the model of BCFNs, and a few current operators such as CF power AOs originated by Hu et al. [39], interval-valued intuitionistic fuzzy (IVIF) PMM operators investigated by Xu et al. [54], BF sine trigonometric (BFST) AOs explored by Riaz et al. [22], and BCF BM operators originated by Mahmood et al. [43]. Now, apply the current and originated operators to the information in Table 3, and the required outcomes are displayed in Tables 6 and Table 7.

Table 6. The outcomes of the prevailing and originated operators.

Source	Operator	$\mathcal{S}(\mathbb{U}_{\mathfrak{AL}-1})$	$\mathcal{S}(\mathbb{U}_{\mathfrak{AL}-2})$	$\mathcal{S}(\mathbb{U}_{\mathfrak{AL}-3})$	$\mathcal{S}(\mathbb{U}_{\mathfrak{AL}-4})$
Hu et al. [39]	**CFPAO**	Disastrous	Disastrous	Disastrous	Disastrous
Xu et al. [54]	**IVIFPMM**	Disastrous	Disastrous	Disastrous	Disastrous
Riaz et al. [22]	**BFSTAO**	Disastrous	Disastrous	Disastrous	Disastrous
Mahmood et al. [43]	**BCFWBM**	0.5905	0.5255	0.5307	0.5909
Originated operator	**BCFPMM**	0.55749	0.4078	0.4978	0.54847
Originated operator	**BCFWPMM**	0.56466	0.40703	0.4935	0.55187
Originated operator	**BCFPDMM**	064756	0.56278	0.56874	0.62041
Originated operator	**BCFWPDMM**	0.64727	0.55579	0.56402	0.62022

Table 7. The ranking among prevailing and originated operators.

Source	Operator	Ranking
Hu et al. [39]	**CFPAO**	Disastrous

Xu et al. [54]	**IVIFPMM**	Disastrous
Riaz et al. [22]	**BFSTAO**	Disastrous
Mahmood et al. [43]	**BCFWBM**	$\mathbb{U}_{\mathfrak{A}\mathfrak{L}-4} > \mathbb{U}_{\mathfrak{A}\mathfrak{L}-1} > \mathbb{U}_{\mathfrak{A}\mathfrak{L}-3} > \mathbb{U}_{\mathfrak{A}\mathfrak{L}-2}$
Originated operator	**BCFPMM**	$\mathbb{U}_{\mathfrak{A}\mathfrak{L}-1} > \mathbb{U}_{\mathfrak{A}\mathfrak{L}-4} > \mathbb{U}_{\mathfrak{A}\mathfrak{L}-3} > \mathbb{U}_{\mathfrak{A}\mathfrak{L}-2}$
Originated operator	**BCFWPMM**	$\mathbb{U}_{\mathfrak{A}\mathfrak{L}-1} > \mathbb{U}_{\mathfrak{A}\mathfrak{L}-4} > \mathbb{U}_{\mathfrak{A}\mathfrak{L}-3} > \mathbb{U}_{\mathfrak{A}\mathfrak{L}-2}$
Originated operator	**BCFPDMM**	$\mathbb{U}_{\mathfrak{A}\mathfrak{L}-1} > \mathbb{U}_{\mathfrak{A}\mathfrak{L}-4} > \mathbb{U}_{\mathfrak{A}\mathfrak{L}-3} > \mathbb{U}_{\mathfrak{A}\mathfrak{L}-2}$
Originated operator	**BCFWPDMM**	$\mathbb{U}_{\mathfrak{A}\mathfrak{L}-1} > \mathbb{U}_{\mathfrak{A}\mathfrak{L}-4} > \mathbb{U}_{\mathfrak{A}\mathfrak{L}-3} > \mathbb{U}_{\mathfrak{A}\mathfrak{L}-2}$

The CF AOs have the ability to provide the weight vector by support measures, but the model of CFS doesn't describe the negative pole and hence can't demonstrate the BCFNs. Similarly, IVIFPMM doesn't describe the 2nd dimension as well as the negative pole, so it can't demonstrate the BCFNs. BF sine trigonometric AOs are modeled for aggregating both polarities that are positive and negative but are not modeled to describe the 2nd dimension, so these operators can't demonstrate the BCFNs. For these reasons, these operators failed to solve the information in Table 1. BCF weighted BM (BCFWBM) originated by Mahmood et al. [43] solved the information in Table 1, and the ranking among the alternatives is demonstrated in Table 6, where we can note that $\mathbb{U}_{\mathfrak{A}\mathfrak{L}-4}$ is the most superb alternative. But all originated operators are giving that. $\mathbb{U}_{\mathfrak{A}\mathfrak{L}-1}$ is the most superb alternative. In the BCFWBM operator, the expert or decision-maker provided the weight vector to each attribute according to his/her choice, which affects the outcomes, so the outcome obtained by utilizing the BCFWBM operator is not fair, and the choice of the expert played an important role. But in the proposed operators, the weighted vector is generated with the assistance of a support measure. Therefore, by employing the originated operator, one can achieve a fair outcome and can choose the most superb alternative. Moreover, in this paper, we also originated an operator where the expert can give a weight vector by his/her choice, which is the BCFWPMM and BBCFWPDM operators.

6. | CONCLUSION

This article addressed gaps identified in the literature by developing a new BCF-MADM method in the context of ERP system selection. The following are the major contributions of this study to the literature and practice of ERP selection. First of all, it is possible to note that we have managed to create a clear and effective approach that takes into account the positive and negative attributes of ERP systems which is a major drawback of the existing approaches. This balanced evaluation gives the decision makers a better and more realistic view of ERP options. Secondly, our method includes additional fuzzy information concerning other attributes of ERP that reflects the stochastic nature and vagueness of the decision process. These additional fuzzy data enrich the evaluation process and make it more reliable with the help of inclusion. Thirdly, new AOs have been developed by adding PA with MM and dual MM in BCFS. These operators, BCFPMM, BCFWPMM, BCFPDMM, and BCFWPDMM, allow for more flexibility in the aggregation process and also capture the interactions between the attribute arguments. The applicability and efficiency of our method are illustrated by applying it to a real-life selection of an ERP system. The reference to the existing approaches indicates the advantage of the proposed method in solving multi-dimensional DM problems. Furthermore, our approach effectively reduces the influence of the extreme values offered by the decision makers, which is a major drawback of MADM techniques. This feature increases the stability and credibility of the decision results.

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