ORIGINAL ARTICLE

OPEN ACCESS

Multi-attribute Group Decision-Making Based on Pythagorean Fuzzy Rough Set and Novel Schweizer-Sklar T-norm and T-conorm

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Article History:

Submitted: August 09, 2022
Revised: November 26, 2022
Accepted: December 11, 2022
Published online: December 30, 2022

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A B S T R A C T

The Pythagorean fuzzy rough set (PyFRS) is a robust framework that plays a vital role in reducing the uncertainty from the extracted information from real-life scenarios. In this article, we proposed some aggregation operators (AOs) based on Schweizer-Sklar t-norm (SSTrM) and Schweizer-Sklar t-conorm (SSTCrM). These AOs include Pythagorean fuzzy rough (PyFR) Schweizer-Sklar weighted averaging (PyFRSSWA) and PyFR Schweizer-Sklar (SS) weighted geometric (PyFRSSWG) operators to deal with the information in the form of PyFR values (PyFRVs). The basic properties of the developed AOs are investigated and then applied to the multi-attribute group decision-making (MAGDM) problem. The variation of the obtained results is obtained by changing the values of the involved parameter in SSTrM and SSTCrM. Additionally, the obtained results are compared with those obtained by existing AOs. Furthermore, all the observations and results are presented graphically.

Keywords: Rough set; fuzzy set; fuzzy rough set, Pythagorean fuzzy rough set; Schweizer-Sklar t-norm; Schweizer-Sklar t-conorm; aggregation operators.

1 | INTRODUCTION

Decision-making (DM) problems are challenging mainly when the decisions are based on multiple factors. The major problem in DM problems is the uncertainty in the information. To handle uncertainty, several techniques and frameworks have been developed. First, the idea of the fuzzy set (FS) was introduced by Zadeh [1] to describe the belongingness of an object to a set with the help of the membership grade (MrGr) $m \in [0,1]$. The FS has been playing an important role to reduce the uncertainty from information. With the introduction of the FS, several real-life scenarios can be described with less hesitation. For example, according to the crisp set, the adultness of a person can be described as either the person is adult or not if the adultness age is 18 years. The question is, what about a person who is 17 years, 11 months, and 29 days old? According to the crisp set, he is not an adult yet but will be an adult after some hours. A lot of such silly things would happen; hence, the FS is the tool that can describe such scenarios by assigning the values to an object from 0 to 1. Another framework as the generalization of the FS, named intuitionistic FS (IFS) was introduced by Atanassov [2] where the belongingness of an object was described with the help of an additional grade i.e., non-membership grade (NMrGr) $u \in [0,1]$. The role of the IFS in reducing the uncertainty from the information is very significant; hence, several researchers have applied the IFS in different scenarios. [3] used IFS in graph theory, [4] used IFS in medical diagnosis, [5] used IFS in the extension of complex IFS, [6] used IFS to introduce intuitionistic fuzzy metric spaces, [7] used IFS in pattern recognition, and so on. To increase the range of the IFS, Yager [8] introduced the Pythagorean FS (PyFS) by changing the condition of the IFS. PyFS became very popular among researchers and has very vast applications. For example, Akram et al. [9] used PyFS in multi-attribute decision-making (MADM), Akram and Shahzadi [10] used PyFS in graphs to solve decision-making problems, Ali et al. [11] used PyFS to solve decision-making problems, and so on.

A rough set (RS) [12] is another tool to reduce uncertainty from information with the help of the lower and upper approximations of the crisp set. RS is also a very useful technique to reduce uncertainty and ambiguity; hence, several researchers used RS in several disciplines. For example, Grzymala-Busse [13] discussed the application of the RS in data mining, and so on. Furthermore, [14] introduced the fuzzy rough set (FRS) idea by making a bridge between the FS and RS. The FRS was an interesting idea where the FS was defined based on the lower and upper approximations for the description of an object. Consequently, the FRS became popular among researchers. For example, [15] used FRS in sample selection, [16] used FRS in neural network selection, [17] introduced topological structures for FRS, [18] used FRS in uncertainty measurement, and so on. As the IFS is the generalization of the FS, the linkage between IFS and RS named the IFRS was introduced by [19] to cover more information than the FRS with less uncertainty. The intuitionistic fuzzy rough (IFR) set (IFRS) is also an essential technique for researchers to reduce the uncertainty of the information. For example, [19] used IFRS to solve the MAGDM problem, [20] used IFRS to solve the MAGDM problem based by using Frank TNrM and TCNrM to operate the IFRVs, [21] used IFRS in surgical instrument selection, and [22] used IFRS to solve DM problem, and so on. Similarly, Mandal and Ranadive [23] introduced PyFRS which is the most genderized idea to reduce the uncertainty at the maximum level.

Several t-norms and t-conorms have been introduced, such as Acze-Alsina [24], Hamacher [25], Frank [26], and so on. These t-norms and t-conorms have played an important role in information handling, especially in fuzzy frameworks. The SSTrM and SSTCrM are very significant generalizations of the t-norm and t-conorm [27]. SSTrM and SSTCrM are very flexible because they involve parameters; hence, SSTrM and SSTCrM provide an opportunity for decision-makers to choose the parameter's value. Due to flexibility, SSTrM and SSTCrM have been widely used by researchers. For example, [28] used SSTrM and SSTCrM to develop a methodology based on Maclaurin symmetric mean operator to solve the MADM problem, [29] used SSTrM and SSTCrM developed AOs based on Pythagorean FS to solve MAGDM problem, [30] used SSTrM and SSTCrM to develop AOs based on T-spherical FS to solve MAGDM problem. Furthermore, the idea of PyFRS is also an important framework to cope with uncertain information, and we found a lack of tools based on SSTrM and SSTCrM for the framework of PyFRS to aggregate information in the form of IFRVs. Hence, this article aims to develop the AOs based on SSTrM and SSTCrM.

The further study consists of four more sections. In section 2, the fundamentals of the articles are stated for better understanding. Section 3 consists of the development of PYFRSSWA and PYFRSSWG operators. In section 3, the basic properties of PYFRSSWA and PYFRSSWG operators are investigated as well. Section 4 includes the application of the developed AOs to the MAGDM problem. In section 4, first, the methodology to use PYFRSSWA and PYFRSSWG operators is discussed and then a real-life example of the MAGDM problem is solved with the help of these operators. The sensitivity and comparative study are also included in section 4. In section 5, the article is summarized.

2 | PRELIMANRIES

This section includes the definitions of the basic terminologies. Some fundamental binary relations, PyFS, PyFRS, PyFRV, SSTrM, and SSTCrM, are defined in this section.

PyFS [8] is an interesting tool to tackle uncertain situations as defined below.

Definition 1: The PyFS $\mathfrak{D}$ on the universe Z is defined as

$$\mathfrak{D} = \{\zeta, (m_\zeta, u_\zeta) : \zeta \in \mathrm{Z}\}$$

The PyFS consists of MrGr u and NMrGr m $\forall\, u_\zeta, m_\zeta : \mathrm{Z} \to [0,1], 0 \le u_\zeta^2 \le 1$. Consider three PyFVs, $\mathfrak{D} = (m_\zeta, u_\zeta)$ and $\mathfrak{D}_{p} = (m_{\zeta p}, u_{\zeta p})$ for $p = 1,2$.Some basic properties of PyFVs are given below.

$$\mathfrak{D}_1 \cup \mathfrak{D}_2 = (\vee(m_1, m_2), \wedge(u_1, u_2))$$

$$\mathfrak{D}_1 \cap \mathfrak{D}_2 = (\wedge(m_1, m_2), \vee(u_1, u_2))$$

$$\mathfrak{D}_1 \oplus \mathfrak{D}_2 = \left(\sqrt{m_1^2 + m_2^2 - m_1^2 m_2^2}, u_1 u_2\right)$$

$$\mathfrak{D}_1 \otimes \mathfrak{D}_2 = \left(m_1 m_2, \sqrt{u_1^2 + u_2^2 - u_1^2 u_2^2}\right)$$

$\mathfrak{D}^c = \left((u)^\lambda, \sqrt{1-(1-m^2)^\lambda}\right)$ where $\mathfrak{D}^c$ is the complement of the PyFV $\mathfrak{D}$

$\sigma\mathfrak{D} = \left(\sqrt{1-(1-u^2)^\lambda}, m^\lambda\right)$ for $\sigma > 0$

The formation of the PyFRS is based on the binary relation [12]. The binary relation Ω is defined as follows.

Definition 2: Let $\Omega \in \mathrm{Z} \times \mathrm{Z}$ be the binary relation on the set Z. Then Ω is

reflexive iff $(\mathcal{M}, \mathcal{M})\ \forall\, \mathcal{M} \in \mathfrak{D}$

symmetric if $(\mathcal{M}, \tau) \in \Omega$ then $(\tau, \mathcal{M}) \in \Omega\ \forall\, \mathcal{M}, \tau \in \mathfrak{D}$

transitive if $\forall\, \mathcal{M}, \tau, \omega \in \mathfrak{D}$ if $(\tau, \omega) \in \Omega$ and $(\omega, \mathcal{M}) \in \Omega$ then $(\tau, \mathcal{M}) \in \Omega$

Definition 3: [12] Let $\mathfrak{D}$ be the universal set and Ω be the relation. Now we assume a mapping $\Omega^*: \mathfrak{D} \to \mathcal{A}(\mathfrak{D})$ as

$$\Omega^*(a) = \{\mathcal{M} \in \mathfrak{D} : (a, \mathcal{M}) \in \Omega\}, for\ a \in \mathfrak{D}$$

Where $\Omega^{*}(a)$ is named as the successor neighborhood of an element a concerning Ω and $(\mathfrak{O},\Omega)$ is said to be the crisp space of the approximation. For any set $\mathfrak{W} \subseteq \mathfrak{O}$ the definition of the LA and UA are stated below.

$$\Omega^{\mathrm{LA}}(\mathfrak{W}) = \{\mathcal{M} \in \Omega | \Omega^{*}(\mathcal{M}) \subseteq \mathfrak{W}\}$$
$$\Omega^{\mathrm{UA}}(\mathfrak{W}) = \{\mathcal{M} \in \Omega | \Omega^{*}(\mathcal{M}) \cap \mathfrak{W} \neq \phi\}$$

The set $\left(\Omega^{\mathrm{LA}}(\mathfrak{W}), \Omega^{\mathrm{UA}}(\mathfrak{W})\right)$ is said to be an RS based on LA and UA.

Definition 4: [12] Let $\mathfrak{O}$ be the universal set and Ω be the relation from $PyFS(\mathfrak{O} \times \mathfrak{O})$. Then

Ω is reflexive iff $m_{\Omega}(\mathcal{M},\mathcal{M}) = 1$ and$u_{\Omega}(\mathcal{M},\mathcal{M}) = 0 \ \forall\, \mathcal{M} \in \mathfrak{O}$

Ω is symmetric if $\forall(\mathcal{M},\tau) \in \mathfrak{O} \times \mathfrak{O}$ then $m_{\Omega}\,(\tau,\mathcal{M}) = m_{\Omega}(\mathcal{M},\tau) \ \forall\, \mathcal{M},\tau \in \mathfrak{O}$ and $u_{\Omega}\,(\tau,\mathcal{M}) = u_{\Omega}(\mathcal{M},\tau)$

Ω is transitive if $\forall\, \mathcal{M},\tau,\omega \in \mathfrak{O}$ if $(\tau,\omega) \in \Omega$ and $(\omega,\mathcal{M}) \in \Omega$ then $m_{\Omega}(\tau,\mathcal{M}) \geq \vee[m_{\Omega}(\tau,\omega) \wedge m_{\Omega}(\omega,\mathcal{M})]$ and $u_{\Omega}(\tau,\mathcal{M}) \geq \wedge[\, u_{\Omega}(\tau,\omega) \wedge u_{\Omega}(\omega,\mathcal{M})]$.

Definition 5: [23] Let $\mathfrak{O}$ be the universal set and $\Omega \in \mathfrak{O} \times \mathfrak{O}$ be the PyF relation then $(\mathfrak{O},\Omega)$ is said to be the PyF space of the approximation. For any set $\mathfrak{W} \subseteq PyFS(\mathfrak{O})$ the definition of the PyFLA and PyFUA are stated below.

$$\Omega^{\mathrm{PyFUA}}(\mathfrak{W}) = \left\{\mathcal{M}, m_{\Omega^{\mathrm{PyFUA}}(\mathfrak{W})}(\mathcal{M}), u_{\Omega^{\mathrm{PyFUA}}(\mathfrak{W})}(\mathcal{M}) | \mathcal{M} \in \mathfrak{O}\right\}$$
$$\Omega^{\mathrm{PyFLA}}(\mathfrak{W}) = \left\{\mathcal{M}, m_{\Omega^{\mathrm{PyFLA}}(\mathfrak{W})}(\mathcal{M}), u_{\Omega^{\mathrm{PyFLA}}(\mathfrak{W})}(\mathcal{M}) | \mathcal{M} \in \mathfrak{O}\right\}$$

Where,

$$m_{\Omega^{\mathrm{PyFUA}}(\mathfrak{W})}(\mathcal{M}) = \bigvee_{l \in \mathfrak{O}} \left[m_{\Omega(\mathcal{M})}(\mathcal{M},l) \vee m_{\mathfrak{W}(\mathcal{M})}\right]$$
$$u_{\Omega^{\mathrm{PyFUA}}(\mathfrak{W})}(\mathcal{M}) = \bigwedge_{l \in \mathfrak{O}} \left[u_{\Omega(\mathcal{M})}(\mathcal{M},l) \wedge u_{\mathfrak{W}(\mathcal{M})}\right]$$
$$m_{\Omega^{\mathrm{PyFLA}}(\mathfrak{W})}(\mathcal{M}) = \bigwedge_{l \in \mathfrak{O}} \left[m_{\Omega(\mathcal{M})}(\mathcal{M},l) \wedge m_{\mathfrak{W}(\mathcal{M})}\right]$$
$$u_{\Omega^{\mathrm{PyFLA}}(\mathfrak{W})}(\mathcal{M}) = \bigvee_{l \in \mathfrak{O}} \left[u_{\Omega(\mathcal{M})}(\mathcal{M},l) \vee u_{\mathfrak{W}(\mathcal{M})}\right]$$

With condition $0 \leq m_{\Omega^{\mathrm{PyFUA}}(\mathfrak{W})}(\mathcal{M}) + u_{\Omega^{\mathrm{PyFUA}}(\mathfrak{W})}(\mathcal{M}) \leq 1$ and $0 \leq m_{\Omega^{\mathrm{PyFLA}}(\mathfrak{W})}(\mathcal{M}) + u_{\Omega^{\mathrm{PyFLA}}(\mathfrak{W})}(\mathcal{M}) \leq 1$. The set $\left(\Omega^{\mathrm{PyFLA}}(\mathfrak{W}), \Omega^{\mathrm{PyFUA}}(\mathfrak{W})\right)$ is said to be a PyFRS based on PyFLA and PyFUA.

Definition 6: [31] The SSTrM and SSTCrM are defined as

$$\Psi_{SS}(o,\varsigma) = (o^{\alpha} + \varsigma^{\alpha} - 1)^{1/\alpha}$$
$$\Psi^{*}{}_{SS}(o,\varsigma) = 1 - ((1-o)^{\alpha} + (1-\varsigma)^{\alpha} - 1)^{1/\alpha}.$$

3 | AGGREGATION OPERATORS FOR PYFRVS BASED ON SSTRM AND SSTCRM

In this section, a family of weighted averaging and geometric AOs is developed by using the SSTrM and SSTCrM. First, some operational laws for PyFRVs are defined based on the SSTrM and SSTCrM to develop these AOs.

Some basic operational laws for PyFRVs are defined below.

Definition 7: Let $\mathfrak{N}_{\wp} = \left((m_{\wp}^{la}, u_{\wp}^{la}), (m_{\wp}^{ua}, u_{\wp}^{ua}) \right), \wp = 1,2$ be two PyFRVs. Then

$$\mathfrak{N}_1 \oplus \mathfrak{N}_2 = \begin{pmatrix} \left(\sqrt{1 - \left(\left(1 - m_1^{2la}\right)^{\alpha} + \left(1 - m_2^{2la}\right)^{\alpha} - 1 \right)^{1/\alpha}}, \left(\left(u_1^{la}\right)^{\alpha} + \left(u_2^{la}\right)^{\alpha} - 1 \right)^{1/\alpha} \right) \\ \left(\sqrt{1 - ((1 - m_1^{2ua})^{\alpha} + (1 - m_2^{2ua})^{\alpha} - 1)^{1/\alpha}}, ((u_1^{ua})^{\alpha} + (u_2^{ua})^{\alpha} - 1)^{1/\alpha} \right) \end{pmatrix}$$

$$\mathfrak{N}_1 \otimes \mathfrak{N}_2 = \begin{pmatrix} \left(\left(\left(m_1^{la}\right)^{\alpha} + \left(m_2^{la}\right)^{\alpha} - 1 \right)^{1/\alpha}, \sqrt{1 - \left(\left(1 - u_1^{2la}\right)^{\alpha} + \left(1 - u_2^{2la}\right)^{\alpha} - 1 \right)^{1/\alpha}} \right) \\ \left(((m_1^{ua})^{\alpha} + (m_2^{ua})^{\alpha} - 1)^{1/\alpha}, \sqrt{1 - ((1 - u_1^{2ua})^{\alpha} + (1 - u_2^{2ua})^{\alpha} - 1)^{1/\alpha}} \right) \end{pmatrix} \tag{1}$$

$$\eta \mathfrak{N}_1 = \begin{pmatrix} \left(\sqrt{1 - \left(\eta \left(1 - m_1^{2la}\right)^{\alpha} - (\eta - 1) \right)^{1/\alpha}}, \left(\eta \left(u_1^{la}\right)^{\alpha} - (\eta - 1) \right)^{1/\alpha} \right) \\ \left(\sqrt{1 - \left(\eta (1 - m_1^{2ua})^{\alpha} - (\eta - 1) \right)^{1/\alpha}}, \left(\eta (u_1^{ua})^{\alpha} - (\eta - 1) \right)^{1/\alpha} \right) \end{pmatrix}$$

$$\mathfrak{N}_1^{\eta} = \begin{pmatrix} \left(\left(\eta \left(m_1^{la}\right)^{\alpha} - (\eta - 1) \right)^{1/\alpha}, \sqrt{1 - \left(\eta \left(1 - u_1^{2la}\right)^{\alpha} - (\eta - 1) \right)^{1/\alpha}} \right) \\ \left(\eta (m_1^{ua})^{\alpha} - (\eta - 1) \right)^{1/\alpha}, \sqrt{1 - \left(\eta (1 - u_1^{2ua})^{\alpha} - (\eta - 1) \right)^{1/\alpha}} \end{pmatrix}$$

Based on these operational laws, weighted averaging AO is defined below.

Definition 8: Let $\mathfrak{N}_{\wp} = \left((m_{\wp}^{la}, u_{\wp}^{la}), (m_{\wp}^{ua}, u_{\wp}^{ua}) \right), \wp = 1,2,\cdots,\mathcal{K}$ are $\mathcal{K}$ PyFRVs and $\mathfrak{W}_{\wp}$ is the weight of the $\wp th$ PyFRV such that $\sum_{\wp=1}^{\mathcal{K}} \mathfrak{W}_{\wp} = 1$ (Note that the notation $\mathfrak{W}_{\wp}$ is used for the weight of $\wp th$ PyFRV unless otherwise stated). Then the PyFRSSWA operators are defined as

$$PyFRSSWA(\mathfrak{N}_1, \mathfrak{N}_2, \cdots, \mathfrak{N}_{\mathcal{K}}) = \bigoplus_{\wp=1}^{\mathcal{K}} \mathfrak{W}_{\wp} \mathfrak{N}_{\wp}$$

$$= \begin{pmatrix} \left(\sqrt{1 - \left(\sum_{\wp=1}^{\mathcal{K}} \mathfrak{W}_{\wp} \left(1 - m_{\wp}^{2la}\right)^{\alpha} \right)^{\frac{1}{\alpha}}}, \left(\sum_{\wp=1}^{\mathcal{K}} \mathfrak{W}_{\wp} \left(u_{\wp}^{la}\right)^{\alpha} \right)^{\frac{1}{\alpha}} \right) \\ \left(\sqrt{1 - \left(\sum_{\wp=1}^{\mathcal{K}} \mathfrak{W}_{\wp} \left(1 - m_{\wp}^{2ua}\right)^{\alpha} \right)^{\frac{1}{\alpha}}}, \left(\sum_{\wp=1}^{\mathcal{K}} \mathfrak{W}_{\wp} \left(u_{\wp}^{ua}\right)^{\alpha} \right)^{\frac{1}{\alpha}} \right) \end{pmatrix} \tag{2}$$

Some basic properties of the PyFRSSWA operator defined in Equation 2 are stated and proven in the following.

Theorem 1: Let $\mathfrak{N}_{\wp} = \left((m_{\wp}^{la}, u_{\wp}^{la}), (m_{\wp}^{ua}, u_{\wp}^{ua}) \right), \wp = 1,2,\cdots,\mathcal{K}$ are $\mathcal{K}$ PyFRVs. Then the PyFRV is obtained after aggregation by the PyFRSSWA operator as follows.

$$PyFRSSWA(\mathfrak{N}_1,\mathfrak{N}_2,\cdots,\mathfrak{N}_{\mathcal{K}})=\left(\begin{array}{c}\left(\sqrt{1-\left(\sum_{\wp=1}^{\mathcal{K}}\mathfrak{W}_\wp\left(1-m_\wp^{2la}\right)^\alpha\right)^{\frac{1}{\alpha}}},\left(\sum_{\wp=1}^{\mathcal{K}}\mathfrak{W}_\wp\left(u_\wp^{la}\right)^\alpha\right)^{\frac{1}{\alpha}}\right)\\ \left(\sqrt{1-\left(\sum_{\wp=1}^{\mathcal{K}}\mathfrak{W}_\wp\left(1-m_\wp^{2ua}\right)^\alpha\right)^{\frac{1}{\alpha}}},\left(\sum_{\wp=1}^{\mathcal{K}}\mathfrak{W}_\wp\left(u_\wp^{ua}\right)^\alpha\right)^{\frac{1}{\alpha}}\right)\end{array}\right)$$

Proof: We apply mathematical induction to prove Theorem 1 as follows.

For $\mathcal{K}=2$

$$PyFRSSWA(\mathfrak{N}_1,\mathfrak{N}_2)=\left(\begin{array}{c}\left(\sqrt{1-\left(\sum_{\wp=1}^{2}\mathfrak{W}_\wp\left(1-m_\wp^{2la}\right)^\alpha\right)^{\frac{1}{\alpha}}},\left(\sum_{\wp=1}^{2}\mathfrak{W}_\wp\left(u_\wp^{la}\right)^\alpha\right)^{\frac{1}{\alpha}}\right)\\ \left(\sqrt{1-\left(\sum_{\wp=1}^{2}\mathfrak{W}_\wp\left(1-m_\wp^{2ua}\right)^\alpha\right)^{\frac{1}{\alpha}}},\left(\sum_{\wp=1}^{2}\mathfrak{W}_\wp\left(u_\wp^{ua}\right)^\alpha\right)^{\frac{1}{\alpha}}\right)\end{array}\right)$$

Which is PyFRV.

Assume Equation 2 is true for $\mathcal{K}=n$

$$PyFRSSWA(\mathfrak{N}_1,\mathfrak{N}_2,\cdots,\mathfrak{N}_n)=\left(\begin{array}{c}\left(\sqrt{1-\left(\sum_{\wp=1}^{n}\mathfrak{W}_\wp\left(1-m_\wp^{2la}\right)^\alpha\right)^{\frac{1}{\alpha}}},\left(\sum_{\wp=1}^{n}\mathfrak{W}_\wp\left(u_\wp^{la}\right)^\alpha\right)^{\frac{1}{\alpha}}\right)\\ \left(\sqrt{1-\left(\sum_{\wp=1}^{n}\mathfrak{W}_\wp\left(1-m_\wp^{2ua}\right)^\alpha\right)^{\frac{1}{\alpha}}},\left(\sum_{\wp=1}^{n}\mathfrak{W}_\wp\left(u_\wp^{ua}\right)^\alpha\right)^{\frac{1}{\alpha}}\right)\end{array}\right)$$

We show Equation 2 true for $\mathcal{K}=n+1$,

$$PyFRSSWA(\mathfrak{N}_1,\mathfrak{N}_2,\cdots,\mathfrak{N}_n,\mathfrak{N}_{n+1})$$

$$=\left(\begin{array}{c}\left(\sqrt{1-\left(\sum_{\wp=1}^{n}\mathfrak{W}_\wp\left(1-m_\wp^{2la}\right)^\alpha+\mathfrak{W}_{\wp+1}\left(1-m_{\wp+1}^{2la}\right)^\alpha\right)^{\frac{1}{\alpha}}},\left(\sum_{\wp=1}^{n}\mathfrak{W}_\wp\left(u_\wp^{la}\right)^\alpha+\mathfrak{W}_{\wp+1}\left(u_{\wp+1}^{la}\right)^\alpha\right)^{\frac{1}{\alpha}}\right)\\ \left(\sqrt{1-\left(\sum_{\wp=1}^{n}\mathfrak{W}_\wp\left(1-m_\wp^{2ua}\right)^\alpha+\mathfrak{W}_{\wp+1}\left(1-m_{\wp+1}^{2la}\right)^\alpha\right)^{\frac{1}{\alpha}}},\left(\sum_{\wp=1}^{n}\mathfrak{W}_\wp\left(u_\wp^{ua}\right)^\alpha+\mathfrak{W}_{\wp+1}\left(u_{\wp+1}^{la}\right)^\alpha\right)^{\frac{1}{\alpha}}\right)\end{array}\right)$$

Then,

$$PyFRSSWA(\mathfrak{R}_1,\mathfrak{R}_2,\cdots,\mathfrak{R}_{n+1})=\left(\begin{array}{c}\left(\sqrt{1-\left(\sum_{\wp=1}^{n+1}\mathfrak{W}_\wp\left(1-m_\wp^{2la}\right)^\alpha\right)^{\frac{1}{\alpha}}},\left(\sum_{\wp=1}^{n+1}\mathfrak{W}_\wp\left(u_\wp^{la}\right)^\alpha\right)^{\frac{1}{\alpha}}\right)\\ \left(\sqrt{1-\left(\sum_{\wp=1}^{n+1}\mathfrak{W}_\wp\left(1-m_\wp^{2ua}\right)^\alpha\right)^{\frac{1}{\alpha}}},\left(\sum_{\wp=1}^{n+1}\mathfrak{W}_\wp\left(u_\wp^{ua}\right)^\alpha\right)^{\frac{1}{\alpha}}\right)\end{array}\right)$$

This completes the proof of Theorem 1.

Theorem 2 proves the idempotency of the PyFRSSWA operator.

Theorem 2: (Idempotency) Let $\mathfrak{R}_\wp=\left((m_\wp^{la},u_\wp^{la}),(m_\wp^{ua},u_\wp^{ua})\right),\wp=1,2,\cdots,\mathcal{K}$ are $\mathcal{K}$ PyFRVs and $\mathfrak{R}_\wp=\left((m_\wp^{la},u_\wp^{la}),(m_\wp^{ua},u_\wp^{ua})\right)=\left((m^{la},u^{la}),(m^{ua},u^{ua})\right)=\mathfrak{R},\forall\wp=1,2,\cdots,\mathcal{K}$. Then

$$PyFRSSWA(\mathfrak{R}_1,\mathfrak{R}_2,\cdots,\mathfrak{R}_{\mathcal{K}})=\left((m^{la},u^{la}),(m^{ua},u^{ua})\right)=\mathfrak{R}$$

Proof: Since $\mathfrak{R}_\wp=\left((m_\wp^{la},u_\wp^{la}),(m_\wp^{ua},u_\wp^{ua})\right)=\left((m^{la},u^{la}),(m^{ua},u^{ua})\right)$, then

$$PyFRSSWA(\mathfrak{R}_1,\mathfrak{R}_2,\cdots,\mathfrak{R}_{\mathcal{K}})=(\mathfrak{R},\mathfrak{R},\cdots,\mathfrak{R})$$

$$=\left(\begin{array}{c}\left(\sqrt{1-\left(\sum_{\wp=1}^{\mathcal{K}}\mathfrak{W}_\wp\left(1-m_\wp^{2la}\right)^\alpha\right)^{\frac{1}{\alpha}}},\left(\sum_{\wp=1}^{\mathcal{K}}\mathfrak{W}_\wp\left(u_\wp^{la}\right)^\alpha\right)^{\frac{1}{\alpha}}\right)\\ \left(\sqrt{1-\left(\sum_{\wp=1}^{\mathcal{K}}\mathfrak{W}_\wp\left(1-m_\wp^{2ua}\right)^\alpha\right)^{\frac{1}{\alpha}}},\left(\sum_{\wp=1}^{\mathcal{K}}\mathfrak{W}_\wp\left(u_\wp^{ua}\right)^\alpha\right)^{\frac{1}{\alpha}}\right)\end{array}\right)$$

$$=\left(\begin{array}{c}\left(\sqrt{1-\left(\sum_{\wp=1}^{\mathcal{K}}\mathfrak{W}_\wp\left(1-m^{2la}\right)^\alpha\right)^{\frac{1}{\alpha}}},\left(\sum_{\wp=1}^{\mathcal{K}}\mathfrak{W}_\wp\left(u^{la}\right)^\alpha\right)^{\frac{1}{\alpha}}\right)\\ \left(\sqrt{1-\left(\sum_{\wp=1}^{\mathcal{K}}\mathfrak{W}_\wp\left(1-m^{2ua}\right)^\alpha\right)^{\frac{1}{\alpha}}},\left(\sum_{\wp=1}^{\mathcal{K}}\mathfrak{W}_\wp\left(u^{ua}\right)^\alpha\right)^{\frac{1}{\alpha}}\right)\end{array}\right)$$

$$=\left((m^{la},u^{la}),(m^{ua},u^{ua})\right)=\mathfrak{R}$$

This completes the proof.

Theorem 3: (Boundedness) Let $\mathfrak{R}_\wp=\left((m_\wp^{la},u_\wp^{la}),(m_\wp^{ua},u_\wp^{ua})\right),\wp=1,2,\cdots,\mathcal{K}$ are $\mathcal{K}$ PyFRVs and $\mathfrak{R}_\wp^s$ and $\mathfrak{R}_\wp^g$ are the smallest and the greatest PyFRV respectively. Then

$$\mathfrak{R}_\wp^s\leq PyFRSSWA(\mathfrak{R}_1,\mathfrak{R}_2,\cdots,\mathfrak{R}_{\mathcal{K}})\leq\mathfrak{R}_\wp^g$$

Theorem 4: (Monotonicity) Let $\mathfrak{N}_{\wp} = \left(\left(m_{\wp}^{la}, u_{\wp}^{la}\right), \left(m_{\wp}^{ua}, u_{\wp}^{ua}\right)\right), \wp = 1,2,\cdots,\mathcal{K}$ and $\mathfrak{N}_{\wp}^{v} = \left(\left(m_{\wp}^{vla}, u_{\wp}^{vla}\right), \left(m_{\wp}^{vua}, u_{\wp}^{vua}\right)\right)$ are two sets of PyFRVs and $\mathfrak{N}_{\wp} \leq \mathfrak{N}_{\wp}^{v}$. Then

$$PyFRSSWA(\mathfrak{N}_1, \mathfrak{N}_2, \cdots, \mathfrak{N}_{\mathcal{K}}) \leq PyFRSSWA(\mathfrak{N}_1^{v}, \mathfrak{N}_2^{v}, \cdots, \mathfrak{N}_{\mathcal{K}}^{v})$$

Equation 2 defines the PyFRSSWA operator based on the weighted average of PyFRVs. Next, we define the PyFRSSWG operator based on the geometric mean of PyFRVs. In the following, Equation 3 defines the PyFRSSWG operator.

Definition 9: Let $\mathfrak{N}_{\wp} = \left(\left(m_{\wp}^{la}, u_{\wp}^{la}\right), \left(m_{\wp}^{ua}, u_{\wp}^{ua}\right)\right), \wp = 1,2,\cdots,\mathcal{K}$ are $\mathcal{K}$ PyFRVs and $\mathfrak{W}_{\wp}$ denotes the weight of the $\wp th$ PyFRV. Then

$$PyFRAAWG(\mathfrak{N}_1, \mathfrak{N}_2, \cdots, \mathfrak{N}_{\mathcal{K}}) = \bigotimes_{\wp=1}^{\mathcal{K}} \mathfrak{N}_{\wp}^{\mathfrak{W}_{\wp}}$$

$$= \left(\begin{array}{c} \left(\left(\sum_{\wp=1}^{\mathcal{K}} \mathfrak{W}_{\wp} \left(m_{\wp}^{la}\right)^{\alpha} \right)^{\frac{1}{\alpha}}, \sqrt{1 - \left(\sum_{\wp=1}^{\mathcal{K}} \mathfrak{W}_{\wp} \left(1 - u_{\wp}^{2la}\right)^{\alpha} \right)^{\frac{1}{\alpha}}} \right) \\ \left(\left(\sum_{\wp=1}^{\mathcal{K}} \mathfrak{W}_{\wp} \left(m_{\wp}^{ua}\right)^{\alpha} \right)^{\frac{1}{\alpha}}, \sqrt{1 - \left(\sum_{\wp=1}^{\mathcal{K}} \mathfrak{W}_{\wp} \left(1 - u_{\wp}^{2ua}\right)^{\alpha} \right)^{\frac{1}{\alpha}}} \right) \end{array} \right) \tag{3}$$

Some basic properties of the PyFRSSWG operator are stated below. The proofs of these properties are excluded because they are simple as proved above.

Theorem 5: Let $\mathfrak{N}_{\wp} = \left(\left(m_{\wp}^{la}, u_{\wp}^{la}\right), \left(m_{\wp}^{ua}, u_{\wp}^{ua}\right)\right), \wp = 1,2,\cdots,\mathcal{K}$ are $\mathcal{K}$ PyFRVs. Then PyFRV is obtained after the aggregation from the PyFRSSWG operator and

$$PyFRAAWG(\mathfrak{N}_1, \mathfrak{N}_2, \cdots, \mathfrak{N}_{\mathcal{K}}) = \left(\begin{array}{c} \left(\left(\sum_{\wp=1}^{\mathcal{K}} \mathfrak{W}_{\wp} \left(m_{\wp}^{la}\right)^{\alpha} \right)^{\frac{1}{\alpha}}, \sqrt{1 - \left(\sum_{\wp=1}^{\mathcal{K}} \mathfrak{W}_{\wp} \left(1 - u_{\wp}^{2la}\right)^{\alpha} \right)^{\frac{1}{\alpha}}} \right) \\ \left(\left(\sum_{\wp=1}^{\mathcal{K}} \mathfrak{W}_{\wp} \left(m_{\wp}^{ua}\right)^{\alpha} \right)^{\frac{1}{\alpha}}, \sqrt{1 - \left(\sum_{\wp=1}^{\mathcal{K}} \mathfrak{W}_{\wp} \left(1 - u_{\wp}^{2ua}\right)^{\alpha} \right)^{\frac{1}{\alpha}}} \right) \end{array} \right)$$

Theorem 6: (Idempotency) Let $\mathfrak{N}_{\wp} = \left(\left(m_{\wp}^{la}, u_{\wp}^{la}\right), \left(m_{\wp}^{ua}, u_{\wp}^{ua}\right)\right), \wp = 1,2,\cdots,\mathcal{K}$ are $\mathcal{K}$ PyFRVs and $\mathfrak{N}_{\wp} = \left(\left(m_{\wp}^{la}, u_{\wp}^{la}\right), \left(m_{\wp}^{ua}, u_{\wp}^{ua}\right)\right) = \left(\left(m^{la}, u^{la}\right), \left(m^{ua}, u^{ua}\right)\right) = \mathfrak{N}, \forall \wp = 1,2,\cdots,\mathcal{K}$. Then

$$PyFRSSWG(\mathfrak{N}_1, \mathfrak{N}_2, \cdots, \mathfrak{N}_{\mathcal{K}}) = \left(\left(m^{la}, u^{la}\right), \left(m^{ua}, u^{ua}\right)\right) = \mathfrak{N}$$

Theorem 7: (Boundedness) Let $\mathfrak{N}_{\wp} = \left(\left(m_{\wp}^{la}, u_{\wp}^{la}\right), \left(m_{\wp}^{ua}, u_{\wp}^{ua}\right)\right), \wp = 1,2,\cdots,\mathcal{K}$ are $\mathcal{K}$ PyFRVs and $\mathfrak{N}_{\wp}^{s}$ and $\mathfrak{N}_{\wp}^{g}$ are the smallest and the greatest PyFRV respectively. Then

$$\mathfrak{N}_{\wp}^{s} \leq PyFRSSWG(\mathfrak{N}_1, \mathfrak{N}_2, \cdots, \mathfrak{N}_{\mathcal{K}}) \leq \mathfrak{N}_{\wp}^{g}$$

Theorem 8: (Monotonicity) Let $\mathfrak{N}_{\wp} = \left(\left(m_{\wp}^{la}, u_{\wp}^{la}\right), \left(m_{\wp}^{ua}, u_{\wp}^{ua}\right)\right), \wp = 1,2,\cdots,\mathcal{K}$ and $\mathfrak{N}_{\wp}^{v} = \left(\left(m_{\wp}^{vla}, u_{\wp}^{vla}\right), \left(m_{\wp}^{vua}, u_{\wp}^{vua}\right)\right)$ are two sets of PyFRVs and $\mathfrak{N}_{\wp} \leq \mathfrak{N}_{\wp}^{v}$. Then

$$PyFRSSWG(\mathfrak{N}_1, \mathfrak{N}_2, \cdots, \mathfrak{N}_{\mathcal{K}}) \leq PyFRSSWG(\mathfrak{N}_1^{v}, \mathfrak{N}_2^{v}, \cdots, \mathfrak{N}_{\mathcal{K}}^{v})$$

4 | A MULTI-ATTRIBUTES DECISION-MAKING MADM APPROACH BASED ON PROPOSED WORK AND APPLICATIONS

The MAGDM is a very significant procedure to select an alternative among the set of desirable alternatives based on the opinions of some experts corresponding to attributes. A group of experts evaluates each alternative based on some common attributes and give an opinion in the form of PyFRV for each alternative corresponding to each attribute. The obtained information in the decision matrices from the experts is aggregated by using their weights. Then the obtained aggregated information is aggregated for the collective aggregated value of each alternative by using the provided weights of the attributes. The MAGDM is a very useful technique in several fields. It is widely used in business, economics, mathematics, engineering, and so on.

Assume $\{\mathfrak{Y}_1, \mathfrak{Y}_2, \cdots, \mathfrak{Y}_r\}$ are r alternatives evaluated based on the attributes $\{\not{f}_1, \not{f}_2, \cdots, \not{f}_r\}$ by $\{\mathcal{Z}_1, \mathcal{Z}_2, \cdots, \mathcal{Z}_h\}$ experts having weights $\mathcal{W}_n \in [0,1]$ $n = 1,2,\cdots,h$ such that $\sum_{n=1}^{h} \mathcal{W}_n = 1$. We want to select the most suitable alternative among $\{\mathfrak{Y}_1, \mathfrak{Y}_2, \cdots, \mathfrak{Y}_r\}$ with the help of the MAGDM. The steps for the selection of an alternative are given below.

Step 1: The experts give an opinion in the form of the PyFRVs such that $\left(\left(m_{\wp}^{la}, u_{\wp}^{la}\right), \left(m_{\wp}^{ua}, u_{\wp}^{ua}\right)\right)$. Usually, there are different two types of attributes. If there exists a cost type attribute in the list of the attributes, we take the compliment of that attribute which is called normalization such that $\left(\left(u_{\wp}^{la}, m_{\wp}^{la}\right), \left(u_{\wp}^{ua}, m_{\wp}^{ua}\right)\right)$.

Step 2: The information in the matrices obtained after the normalization is ready to aggregate. We utilize PyFRVSSWA/PyFRSSWG operator to aggregate the attribute individually. The aggregated decision matrix is obtained as the result that has the information in the form of the PyFRVs for each alternative corresponding to each attribute.

Step 3: We utilize PyFRSSWA/PyFRSSWG operator to aggregate the attributes collectively for each alternative. We obtain the collective aggregated value in the form of the PyFRVs after the aggregation in this step.

Step 4: Utilize the definition of the score function to find the score of each alternative.

Step 5: At last, we rank the alternatives with the help of the score values obtained for each alternative.

Example 1: Consider the company that wants to invest in the foreign stock exchange. But, investing, the company decides to evaluate possible risks in that sector. The company gives the responsibility to evaluate the possible risks to three experts E_t ($t = 1,2,3$) with weights $(0.35,0.33,0.32)^T$. Consider there are four possible risks $\mathfrak{S}_t(t = 1,2,3,4)$ where

Risk of commodities $\mathfrak{S}_1$: The companies usually make a profit when the costs of the commodities fall and vice versa. Hence, the cost of the commodities is an important factor in the foreign exchange business.

Legislative risk $\mathfrak{S}_2$: The relations between the company and the government are very important for the business. The policies and the rules made by the government affect businesses badly.

Interest rate risk $\mathfrak{S}_3$: The variation in the interest rates also affects the investment in the stock exchange. Hence, the investor should keep an eye on the variation in the interest rates.

Model risk $\mathfrak{S}_4$: The formation of the business and economic model for investment is very important. The wrong model for investment is a waste of investment and time. The risk factors are evaluated based on the attributes with weights $(0.21,0.29,0.23,0.27)^T$. The risk factors mentioned above are evaluated based on the following attributes.

Previous record $\mathcal{F}_1$: Each risk factor is assessed based on its previous record history. Because the previous record is very helpful to understand the behavior of that risk factor.

Expected period $\mathcal{F}_2$: Expected period for any risk is also very important and should be considered.

Expected rate of change $\mathcal{F}_3$: The expected rate of change is also an important attribute that affects the risk factors.

Comparison with companies running in profit $\mathcal{F}_4$: Each risk factor is compared with some existing companies running in profit.

Each alternative is evaluated based on the above-mentioned attributes by the experts. The experts provide the information in the form of the PyFRVs. In the following, Tables 1,2, and 3 show the obtained information from the experts.

Table 1: PyFRVs provided by expert e_1

	$\mathfrak{S}_1$				$\mathfrak{S}_2$				$\mathfrak{S}_3$				$\mathfrak{S}_4$			
	m^{la}	m^{ua}	u^{la}	u^{ua}	m^{la}	m^{ua}	u^{la}	u^{ua}	m^{la}	m^{ua}	u^{la}	u^{ua}	m^{la}	m^{ua}	u^{la}	u^{ua}
$\mathcal{F}_1$	0.42	0.38	0.31	0.52	0.44	0.22	0.43	0.48	0.43	0.25	0.25	0.22	0.28	0.42	0.32	0.35
$\mathcal{F}_2$	0.33	0.34	0.25	0.33	0.33	0.44	0.32	0.22	0.25	0.22	0.44	0.54	0.31	0.22	0.25	0.33
$\mathcal{F}_3$	0.44	0.33	0.22	0.33	0.32	0.38	0.22	0.36	0.44	0.44	0.45	0.44	0.35	0.44	0.22	0.22
$\mathcal{F}_4$	0.22	0.22	0.39	0.28	0.22	0.30	0.28	0.31	0.22	0.33	0.25	0.29	0.32	0.42	0.32	0.35

Table 2: PyFRVs provided by expert e_2

	$\mathfrak{S}_1$				$\mathfrak{S}_2$				$\mathfrak{S}_3$				$\mathfrak{S}_4$			
	m^{la}	m^{ua}	u^{la}	u^{ua}	m^{la}	m^{ua}	u^{la}	u^{ua}	m^{la}	m^{ua}	u^{la}	u^{ua}	m^{la}	m^{ua}	u^{la}	u^{ua}
$\mathcal{F}_1$	0.44	0.22	0.42	0.21	0.42	0.22	0.32	0.41	0.33	0.41	0.35	0.31	0.33	0.41	0.22	0.35
$\mathcal{F}_2$	0.22	0.44	0.42	0.22	0.41	0.33	0.31	0.44	0.28	0.21	0.33	0.33	0.44	0.33	0.31	0.22
$\mathcal{F}_3$	0.41	0.28	0.28	0.31	0.31	0.44	0.44	0.28	0.44	0.21	0.22	0.43	0.32	0.43	0.42	0.35
$\mathcal{F}_4$	0.22	0.41	0.41	0.31	0.21	0.29	0.39	0.44	0.23	0.33	0.21	0.43	0.35	0.35	0.31	0.41

Table 3: PyFRVs provided by expert e_3

	$\mathfrak{S}_1$				$\mathfrak{S}_2$				$\mathfrak{S}_3$				$\mathfrak{S}_4$			
	m^{la}	m^{ua}	u^{la}	u^{ua}	m^{la}	m^{ua}	u^{la}	u^{ua}	m^{la}	m^{ua}	u^{la}	u^{ua}	m^{la}	m^{ua}	u^{la}	u^{ua}
$\mathcal{F}_1$	0.33	0.33	0.42	0.42	0.33	0.44	0.32	0.33	0.29	0.22	0.42	0.35	0.46	0.32	0.38	0.22
$\mathcal{F}_2$	0.22	0.44	0.33	0.33	0.42	0.41	0.21	0.22	0.42	0.44	0.45	0.42	0.33	0.41	0.41	0.33
$\mathcal{F}_3$	0.41	0.42	0.42	0.42	0.22	0.22	0.31	0.41	0.38	0.41	0.41	0.22	0.27	0.36	0.33	0.44
$\mathcal{F}_4$	0.29	0.44	0.25	0.29	0.33	0.33	0.44	0.34	0.41	0.41	0.39	0.44	0.22	0.44	0.36	0.41

Step 1,2: We always normalize the decision matrices in the case of the existence of the cost attribute. In Tables 1,2 and 3, there does not exist any cost type attribute. Hence, we do not need normalization. So, we utilize the PyFRSSWA and PyFRSSWG operators to aggregate the attributes individually for each risk factor. The individual aggregated values of attributes from PyFRSSWA and PyFRSSWG operators are given in Tables 4 and 5 respectively in the following.

Table 4: Individual aggregated values by PyFRSSWA operator

	$\mathfrak{S}_1$				$\mathfrak{S}_2$				$\mathfrak{S}_3$				$\mathfrak{S}_4$			
	m^{la}	m^{ua}	u^{la}	u^{ua}	m^{la}	m^{ua}	u^{la}	u^{ua}	m^{la}	m^{ua}	u^{la}	u^{ua}	m^{la}	m^{ua}	u^{la}	u^{ua}
$\mathcal{F}_1$	0.39	0.31	0.38	0.42	0.32	0.29	0.36	0.41	0.35	0.29	0.27	0.36	0.35	0.38	0.31	0.31
$\mathcal{F}_2$	0.26	0.40	0.34	0.30	0.42	0.39	0.28	0.32	0.31	0.29	0.40	0.44	0.36	0.32	0.33	0.30
$\mathcal{F}_3$	0.44	0.34	0.32	0.35	0.30	0.35	0.34	0.35	0.42	0.36	0.43	0.43	0.31	0.41	0.34	0.35
$\mathcal{F}_4$	0.24	0.36	0.36	0.26	0.28	0.30	0.37	0.37	0.29	0.35	0.31	0.31	0.30	0.40	0.33	0.45

Table 5: Individual aggregated values by PyFRSSWG operator

	$\mathfrak{S}_1$				$\mathfrak{S}_2$				$\mathfrak{S}_3$				$\mathfrak{S}_4$			
	m^{la}	m^{ua}	u^{la}	u^{ua}	m^{la}	m^{ua}	u^{la}	u^{ua}	m^{la}	m^{ua}	u^{la}	u^{ua}	m^{la}	m^{ua}	u^{la}	u^{ua}
$\mathcal{F}_1$	0.40	0.32	0.38	0.39	0.40	0.32	0.35	0.40	0.36	0.31	0.34	0.29	0.37	0.38	0.31	0.31
$\mathcal{F}_2$	0.26	0.41	0.33	0.29	0.38	0.39	0.28	0.30	0.33	0.32	0.40	0.43	0.36	0.33	0.32	0.29
$\mathcal{F}_3$	0.42	0.35	0.30	0.35	0.29	0.37	0.32	0.35	0.42	0.38	0.36	0.37	0.31	0.41	0.32	0.33
$\mathcal{F}_4$	0.24	0.32	0.35	0.29	0.26	0.30	0.37	0.36	0.31	0.35	0.28	0.38	0.30	0.40	0.32	0.38

Step 3: Tables 4 and 5 show the individual aggregated values of the attributes. Now, we need to aggregate the values of the attributes collectively for each risk factor with the help of the PyFRSSWA and PyFRSSWG operators. Tables 6 and 7 show the collective aggregated values of the attributes obtained from PyFRSSWA and PyFRSSWG operators respectively in the following.

Table 6: Collective aggregated values obtained by PyFRSSWA operator

	$\mathfrak{S}_1$				$\mathfrak{S}_2$				$\mathfrak{S}_3$				$\mathfrak{S}_4$			
	m^{la}	m^{ua}	u^{la}	u^{ua}	m^{la}	m^{ua}	u^{la}	u^{ua}	m^{la}	m^{ua}	u^{la}	u^{ua}	m^{la}	m^{ua}	u^{la}	u^{ua}
	0.286	0.333	0.482	0.481	0.316	0.312	0.481	0.495	0.304	0.297	0.508	0.523	0.308	0.342	0.473	0.484

Table 7: Collective aggregated values obtained by the PyFRSSWG operator

$\mathfrak{S}_1$				$\mathfrak{S}_2$				$\mathfrak{S}_3$				$\mathfrak{S}_4$			
m^{la}	m^{ua}	u^{la}	u^{ua}	m^{la}	m^{ua}	u^{la}	u^{ua}	m^{la}	m^{ua}	u^{la}	u^{ua}	m^{la}	m^{ua}	u^{la}	u^{ua}
0.493	0.489	0.321	0.299	0.468	0.491	0.307	0.3324	0.501	0.488	0.321	0.347	0.473	0.296	0.515	0.307

Step 4: Tables 6 and 7 show the collective aggregated values obtained from PyFRSSWA and PyFRSSWG operators. Based on the information provided in Tables 6 and 7, we calculate the score values provided in Table 8 as follows.

Table 8: Score values obtained by PyFRSSWA and PYFRSSWG operators

Operator	Score Values
PyFRSSWA	$u(\mathfrak{S}_1) = -0.067, u(\mathfrak{S}_2) = -0.069, u(\mathfrak{S}_3) = -0.087, u(\mathfrak{S}_4) = -0.061$
PyFRSSWG	$u(\mathfrak{S}_1) = 0.072, u(\mathfrak{S}_2) = 0.065, u(\mathfrak{S}_3) = 0.066, u(\mathfrak{S}_4) = 0.076$

Step 5: Based on the score values obtained in Table 8, we find the ranking of the risk factors and assess the riskiest factor among the foreign stock market. The obtained ranking is provided in Table 9.

Table 9: Ranking of risk factors

Operator	Ranking
PyFRSSWA	$\mathfrak{S}_4 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2 \succ \mathfrak{S}_3$
PyFRSSWG	$\mathfrak{S}_4 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2 \succ \mathfrak{S}_3$

The riskiest factors evaluated by using the PyFRSSWA and PyFRSSWG operators are $\mathfrak{S}_4$ and $\mathfrak{S}_4$ respectively. The PyFRSWG operator is based on the geometric average of the information hence, the PyFRSSWG operator is more reliable than the PyFRSSSWA operator.

4.1 | SENSITIVITY ANALYSIS

As it is clear that the SSTrM and SSTCrM consist of the parameter α due to which they become very flexible and significant in handling fuzzy information. In our example, the value of parameter $\alpha = 3$ is used. However, the rank of the risk factors may be changed with the variation of the parameter. Hence, we study the possible variation in the results obtained from both operators. In the following, Table 10 shows the variation of the ranking in the results obtained from PyFRSSWA and PyFRSSWG operators with the possible variation of the parameter α.

Table 10: Sensitivity analysis of PyFRSSWA and PyFRSSWG operators

α	PyFRSSWA	PyFRSSWG
2	$\mathfrak{S}_4 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2 \succ \mathfrak{S}_3$	$\mathfrak{S}_1 \succ \mathfrak{S}_4 \succ \mathfrak{S}_3 \succ \mathfrak{S}_2$
3	$\mathfrak{S}_4 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2 \succ \mathfrak{S}_3$	$\mathfrak{S}_4 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2 \succ \mathfrak{S}_3$
4	$\mathfrak{S}_4 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2 \succ \mathfrak{S}_3$	$\mathfrak{S}_1 \succ \mathfrak{S}_4 \succ \mathfrak{S}_3 \succ \mathfrak{S}_2$
5	$\mathfrak{S}_4 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2 \succ \mathfrak{S}_3$	$\mathfrak{S}_1 \succ \mathfrak{S}_4 \succ \mathfrak{S}_3 \succ \mathfrak{S}_2$
6	$\mathfrak{S}_4 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2 \succ \mathfrak{S}_3$	$\mathfrak{S}_1 \succ \mathfrak{S}_4 \succ \mathfrak{S}_3 \succ \mathfrak{S}_2$
7	$\mathfrak{S}_4 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2 \succ \mathfrak{S}_3$	$\mathfrak{S}_1 \succ \mathfrak{S}_4 \succ \mathfrak{S}_3 \succ \mathfrak{S}_2$
8	$\mathfrak{S}_4 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2 \succ \mathfrak{S}_3$	$\mathfrak{S}_1 \succ \mathfrak{S}_4 \succ \mathfrak{S}_3 \succ \mathfrak{S}_2$

9	$\mathfrak{S}_4 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2 \succ \mathfrak{S}_3$	$\mathfrak{S}_1 \succ \mathfrak{S}_4 \succ \mathfrak{S}_3 \succ \mathfrak{S}_2$
10	$\mathfrak{S}_4 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2 \succ \mathfrak{S}_3$	$\mathfrak{S}_1 \succ \mathfrak{S}_4 \succ \mathfrak{S}_3 \succ \mathfrak{S}_2$
15	$\mathfrak{S}_4 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2 \succ \mathfrak{S}_3$	$\mathfrak{S}_1 \succ \mathfrak{S}_4 \succ \mathfrak{S}_3 \succ \mathfrak{S}_2$
30	$\mathfrak{S}_4 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2 \succ \mathfrak{S}_3$	$\mathfrak{S}_1 \succ \mathfrak{S}_4 \succ \mathfrak{S}_3 \succ \mathfrak{S}_2$
40	$\mathfrak{S}_4 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2 \succ \mathfrak{S}_3$	$\mathfrak{S}_1 \succ \mathfrak{S}_4 \succ \mathfrak{S}_3 \succ \mathfrak{S}_2$
50	$\mathfrak{S}_4 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2 \succ \mathfrak{S}_3$	$\mathfrak{S}_1 \succ \mathfrak{S}_4 \succ \mathfrak{S}_3 \succ \mathfrak{S}_2$

As is clear from Table 10, the riskiest factor obtained from the PyFRSSWA and PyFRSSWG operators are $\mathfrak{S}_4$ and $\mathfrak{S}_1$ respectively at $\alpha = 3$. However, we change the values of α and observe the ranking at the different values of α for both PyFRSSWA and PyFRSSWG operators. Interestingly, we get the same ranking for each value of α. For better understanding, we plotted the sensitivity observation of PyFRSSWA and PyFRSSWG operators in Figures 1 and 2 respectively. The ranking obtained by the PyFRSSWA operator at the different values of α is plotted in Figure 1 as follows.

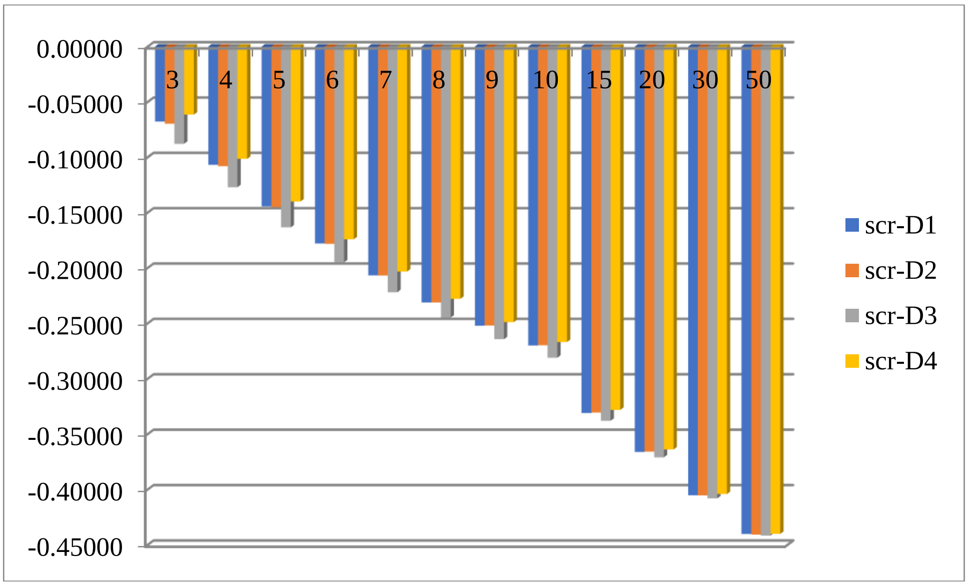

Figure 1: Sensitivity analysis of PYFRSSWA operator

Figure 1 shows the ranking of the risk factors of the foreign stock market at the different values of α. According to Figure 1, the riskiest factor for all values of α is $\mathfrak{S}_4$. However, it is also observed that with the increasing value of α the PyFRSSWA gives nearly equal score values and there is difficulty to distinguish between risk factors which are limitations of the PyFRSSWA operator.

The ranking obtained from the PyFRSSWG operator at the different values of α is plotted in Figure 2 as follows.

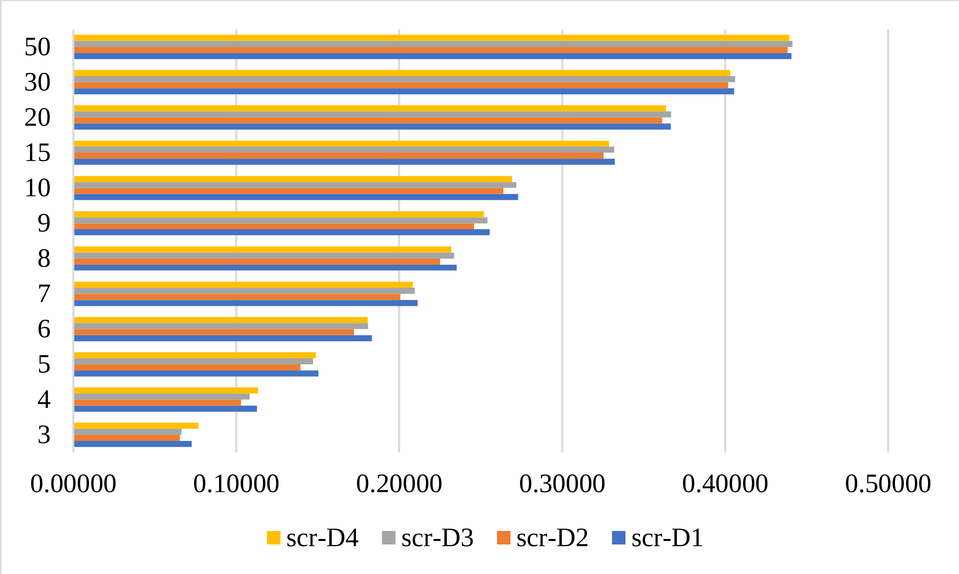

Figure 2: Sensitivity analysis of IRAAWG operator

It is clear from Figure 2 that the ranking obtained from the PyFRSSWG operator remains the same at all the values of α. However, some different rankings of the risk factors are observed at the values of α greater than 20. Hence, we recommend the decision maker use the value $\alpha \leq 20$ for better and fair results.

4.2 | COMPARATIVE ANALYSIS

[20] developed the IFR weighted averaging (IFRWA) and IFR weighted geometric (IFRWG) operators and applied them to solve the MAGDM problem. Further, [19] developed IFR Frank weighted averaging (IFRFWA) and geometric (IFRFWG) operators and solved the MAGDM problem. Some other contributions towards the solution of the MAGDM problem are made by [5], [32], and [33]. In the following, Table 11 shows the comparison between the AOs defined in[19], [20], [5], [32], and [33] with the developed PyFRSSWA and PyFRSSWG operators.

Table 11: Comparative study of PyFRSSWA and PyFRSSWG operators

Operator	Score Values
PyFRSSWA	$\mathfrak{S}_4 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2 \succ \mathfrak{S}_3$
PyFRSSWG	$\mathfrak{S}_4 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2 \succ \mathfrak{S}_3$
IFFRWA	Failed
IFRWG	Failed
IFRFWA	Failed
IFRFWG	Failed
AOs in [5]	Not applicable
AOs in [32]	Not applicable
AOs in [33]	Not applicable

It is clear from Table 11 that the riskiest factor obtained from all the developed averaging operators is $\mathfrak{S}_4$. Additionally, the riskiest factor obtained from all the developed geometric AOs is $\mathfrak{S}_4$. The IFRWA, IFRWG, IFRFWA, and IFRFWG operators cannot deal with the information in the form of the PyFRVs. It is also noted that the AOs introduced in [5], [32], and [33] are unable to entertain the information in the form of the PyFRVs. In the following, Figure 3 is plotted for a better understanding of the comparison of existing AOs with the PyFRSSWA and PyFRSSWG operators respectively.

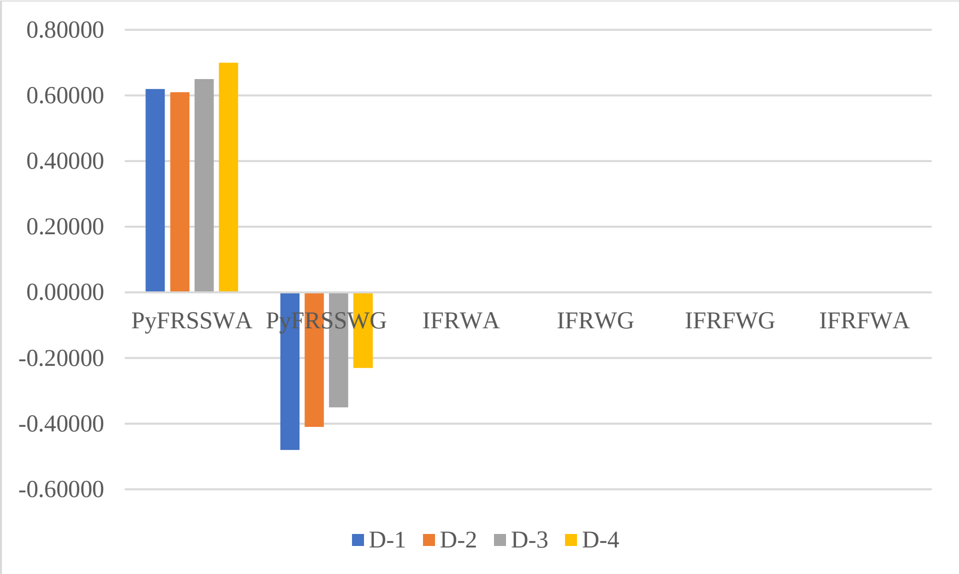

Figure 3: Comparison of existing AOs with PyFRSSWA and PyFRSSWG operators

5 | CONCLUSION

In this article, PyFRSSWA and PyFRSSWG operators have been developed to entertain the information in the form of PyFRVs based on SSTrM and SSTCrM. Some basic properties of the developed AOs have been investigated. The problem of MAGDM has been solved with the help of the developed AOs. In addition, the obtained results are observed at the different values of the parameter involved in SSTrM and SSTCrM and then results have also been compared with the existing techniques for significance. The developed PyFRSSWA and PyFRSSWG operators have been found significant because the SSTrM and SSTCrM are very flexible operational laws for fuzzy information. SSTrM and SSTCrM provide the flexibility to the decision makers to select the value of the parameter of their own choice. Furthermore, the PyFRS is the framework that is the bridge between PyFS and the RS and hence is a very significant technique to solve decision-making problems. PyFRS aggregates the information at the initial stages and then the information in the form of the PyFRVs is further aggregated by the PyFRSSWA and PyFRSSWG operators. Consequently, the results obtained are more optimized and reliable than the other operators and frameworks. In the future, we aim to extend the developed approach to the framework defined in [34]. The developed method can also be applied to the framework defined in [35], [36].

Acknowledgment: We are thankful to the editors and anonymous reviewers for their constructive suggestions. All authors contributed equally to this manuscript.

Declaration of Conflicting Interest: The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding: The author(s) received no financial support for the research, authorship, and/or publication of this article.

REFERENCES

[1] L. A. Zadeh, "Fuzzy sets," *Inf. Control*, vol. 8, no. 3, pp. 338–353, Jun. 1965, doi: 10.1016/S0019-9958(65)90241-X.
[2] K. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets Syst.*, vol. 20, no. 1, pp. 87–96, 1986.
[3] N. Alshehri and M. Akram, "Intuitionistic fuzzy planar graphs," *Discrete Dyn. Nat. Soc.*, vol. 2014, 2014.
[4] S. K. De, R. Biswas, and A. R. Roy, "An application of intuitionistic fuzzy sets in medical diagnosis," *Fuzzy Sets Syst.*, vol. 117, no. 2, pp. 209–213, Jan. 2001, doi: 10.1016/S0165-0114(98)00235-8.
[5] R. Ali, S. Abdullah, S. Muhammad, M. Naeem, and R. Chinram, "Complex intuitionistic fuzzy Maclaurin symmetric mean operators and its application to emergency program selection," *J. Intell. Fuzzy Syst.*, vol. 41, no. 1, pp. 517–538, 2021.
[6] C. Alaca, D. Turkoglu, and C. Yildiz, "Fixed points in intuitionistic fuzzy metric spaces," *Chaos Solitons Fractals*, vol. 29, no. 5, pp. 1073–1078, Sep. 2006, doi: 10.1016/j.chaos.2005.08.066.
[7] C.-H. Chu, K.-C. Hung, and P. Julian, "A complete pattern recognition approach under Atanassov's intuitionistic fuzzy sets," *Knowl.-Based Syst.*, vol. 66, pp. 36–45, 2014.
[8] R. R. Yager, "Pythagorean fuzzy subsets," in *2013 joint IFSA world congress and NAFIPS annual meeting (IFSA/NAFIPS)*, IEEE, 2013, pp. 57–61.
[9] M. Akram, W. A. Dudek, and J. M. Dar, "Pythagorean Dombi fuzzy aggregation operators with application in multicriteria decision-making," *Int. J. Intell. Syst.*, vol. 34, no. 11, pp. 3000–3019, 2019.
[10] M. Akram and G. Shahzadi, "Decision-making approach based on Pythagorean Dombi fuzzy soft graphs," *Granul. Comput.*, vol. 6, no. 3, pp. 671–689, 2021.
[11] Z. Ali, T. Mahmood, K. Ullah, and Q. Khan, "Einstein Geometric Aggregation Operators using a Novel Complex Interval-valued Pythagorean Fuzzy Setting with Application in Green Supplier Chain Management," *Rep. Mech. Eng.*, vol. 2, no. 1, pp. 105–134, 2021.
[12] Z. Pawlak, "Rough sets," *Int. J. Comput. Inf. Sci.*, vol. 11, no. 5, pp. 341–356, Oct. 1982, doi: 10.1007/BF01001956.
[13] J. W. Grzymala-Busse, "Rough Set Theory with Applications to Data Mining," in *Real World Applications of Computational Intelligence*, M. Gh. Negoita and B. Reusch, Eds., in Studies in Fuzziness and Soft Computing, vol. 179. Berlin, Heidelberg: Springer Berlin Heidelberg, 2005, pp. 221–244. doi: 10.1007/11364160_7.
[14] D. DUBOIS and H. PRADE, "Rough Fuzzy Sets and Fuzzy Rough Sets*," *Int. J. Gen. Syst.*, vol. 17, no. 2–3, pp. 191–209, Jun. 1990, doi: 10.1080/03081079008935107.
[15] A. Hadrani, K. Guennoun, R. Saadane, and M. Wahbi, "Fuzzy rough sets: Survey and proposal of an enhanced knowledge representation model based on automatic noisy sample detection," *Cogn. Syst. Res.*, vol. 64, pp. 37–56, Dec. 2020, doi: 10.1016/j.cogsys.2020.05.001.
[16] W. Ji *et al.*, "Fuzzy rough sets and fuzzy rough neural networks for feature selection: A review," *WIREs Data Min. Knowl. Discov.*, vol. 11, no. 3, p. e1402, 2021, doi: 10.1002/widm.1402.
[17] W. Tang, J. Wu, and D. Zheng, "On Fuzzy Rough Sets and Their Topological Structures," *Math. Probl. Eng.*, vol. 2014, p. e546372, Jan. 2014, doi: 10.1155/2014/546372.
[18] I. Düntsch and G. Gediga, "Uncertainty measures of rough set prediction," *Artif. Intell.*, vol. 106, no. 1, pp. 109–137, Nov. 1998, doi: 10.1016/S0004-3702(98)00091-5.

[19] R. Chinram, A. Hussain, T. Mahmood, and M. I. Ali, "EDAS Method for Multi-Criteria Group Decision Making Based on Intuitionistic Fuzzy Rough Aggregation Operators," *IEEE Access*, vol. 9, pp. 10199–10216, 2021, doi: 10.1109/ACCESS.2021.3049605.
[20] M. Yahya, M. Naeem, S. Abdullah, M. Qiyas, and M. Aamir, "A Novel Approach on the Intuitionistic Fuzzy Rough Frank Aggregation Operator-Based EDAS Method for Multicriteria Group Decision-Making," *Complexity*, vol. 2021, p. e5534381, Jun. 2021, doi: 10.1155/2021/5534381.
[21] T. Zhou and J. P. Wachs, "Needle in a haystack: Interactive surgical instrument recognition through perception and manipulation," *Robot. Auton. Syst.*, vol. 97, pp. 182–192, Nov. 2017, doi: 10.1016/j.robot.2017.08.013.
[22] H. Zhang, L. Shu, and S. Liao, "Intuitionistic Fuzzy Soft Rough Set and Its Application in Decision Making," *Abstr. Appl. Anal.*, vol. 2014, p. e287314, May 2014, doi: 10.1155/2014/287314.
[23] P. Mandal and A. S. Ranadive, "Decision-theoretic rough sets under Pythagorean fuzzy information," *Int. J. Intell. Syst.*, vol. 33, no. 4, pp. 818–835, 2018, doi: 10.1002/int.21969.
[24] J. Aczél and C. Alsina, "Characterizations of some classes of quasilinear functions with applications to triangular norms and to synthesizing judgements," *Aequationes Math.*, vol. 25, no. 1, pp. 313–315, Dec. 1982, doi: 10.1007/BF02189626.
[25] H. Garg, "Intuitionistic fuzzy hamacher aggregation operators with entropy weight and their applications to multi-criteria decision-making problems," *Iran. J. Sci. Technol. Trans. Electr. Eng.*, vol. 43, no. 3, pp. 597–613, 2019.
[26] M. J. Frank, "On the simultaneous associativity ofF (x, y) andx+ y- F (x, y)," *Aequationes Math.*, vol. 19, no. 1, pp. 194–226, 1979.
[27] D. Butnariu and E. P. Klement, "Triangular norm-based measures and their Markov kernel representation," *J. Math. Anal. Appl.*, vol. 162, no. 1, pp. 111–143, Nov. 1991, doi: 10.1016/0022-247X(91)90181-X.
[28] P. Wang and P. Liu, "Some Maclaurin symmetric mean aggregation operators based on Schweizer-Sklar operations for intuitionistic fuzzy numbers and their application to decision making," *J. Intell. Fuzzy Syst.*, vol. 36, no. 4, pp. 3801–3824, Jan. 2019, doi: 10.3233/JIFS-18801.
[29] A. Biswas and N. Deb, "Pythagorean fuzzy Schweizer and Sklar power aggregation operators for solving multi-attribute decision-making problems," *Granul. Comput.*, vol. 6, no. 4, pp. 991–1007, Oct. 2021, doi: 10.1007/s41066-020-00243-1.
[30] Q. Khan, J. Gwak, M. Shahzad, and M. K. Alam, "A novel approached based on T-spherical fuzzy Schweizer-Sklar power Heronian mean operator for evaluating water reuse applications under uncertainty," *Sustainability*, vol. 13, no. 13, p. 7108, 2021.
[31] S. B, "Associative funcrions and statistical triangle inequalities," *Puble Math Debr.*, vol. 8, pp. 169–186, 1961.
[32] H. Garg and K. Kumar, "Some Aggregation Operators for Linguistic Intuitionistic Fuzzy Set and its Application to Group Decision-Making Process Using the Set Pair Analysis," *Arab. J. Sci. Eng.*, vol. 43, no. 6, pp. 3213–3227, Jun. 2018, doi: 10.1007/s13369-017-2986-0.
[33] P. Liu, T. Mahmood, and Q. Khan, "Multi-attribute decision-making based on prioritized aggregation operator under hesitant intuitionistic fuzzy linguistic environment," *Symmetry*, vol. 9, no. 11, p. 270, 2017.
[34] T. Mahmood, "A novel approach towards bipolar soft sets and their applications," *J. Math.*, vol. 2020, 2020.
[35] M. Riaz and M. R. Hashmi, "Linear Diophantine fuzzy set and its applications towards multi-attribute decision-making problems," *J. Intell. Fuzzy Syst.*, vol. 37, no. 4, pp. 5417–5439, 2019.
[36] T. Senapati and G. Chen, "Some novel interval-valued Pythagorean fuzzy aggregation operator based on Hamacher triangular norms and their application in MADM issues," *Comput. Appl. Math.*, vol. 40, no. 4, pp. 1–27, 2021.